9 WORKING WITH "PARTS OF A WHOLE" AND OTHER MEANINGS OF RATIONAL NUMBERS AND COMMON FRACTIONS

TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 9•1: Rational Numbers

Many adult students do not explicitly understand the distinction between discrete quantities (collections of separate, indivisible, and thus, countable items) and continuous quantities (uninterrupted entities that require measuring for quantification). **Probe 9.1: Discrete Versus Continuous Quantities** (page 9-3 of the *Student Guide*) was designed to help students explicitly distinguish between these two types of quantity. This knowledge is important for understanding the need for fractions (and measurement) and their nature.

Many adult students also do not explicitly recognize that equal partitioning of a whole is a fundamental conceptual basis for understanding fractions (and measurement). **Part I of Investigation 9.1: Understanding the Conceptual Basis for Fractions** (page 9-4 of the *Student Guide*) illustrates how this process enabled humankind to describe a portion of something (a part of a whole). **Part II** (page 9-5) involves relating equal partitioning to a fair-sharing problem involving a continuous quantity. This serves to introduce what may be a novel idea for some students: Fractions can represent division (e.g., fair-sharing) situations as well as a part-of-the-whole meaning. **Part III** takes Part II a step further and illustrates the mathematical reason for the rational numbers: Unlike other operations, the division of whole numbers does not always result in a whole number (e.g., 10 ÷ 3). In situations involving the division of discrete quantities (Problem B), leftover items cannot be subdivided, so using a remainder notation makes sense (e.g., 10 steelies shared fairly among three boys would result in each getting 3 marbles with one leftover: 10 ÷ 3 = 3 r1). In situations involving continuous quantities (Problem D), leftover items can be subdivided, so a representation for a part of a whole is needed (e.g., 10 cookies shared fairly among four children results in each getting $2\frac{1}{2}$ cookies). For the great many students who simply memorized mathematical definitions and procedures in elementary and high school, this probe can be the first time they have reflected on and understood why division situations are sometimes written with a remainder notation and other times written with a fractional notation.

Furthermore, many students do not explicitly recognize that fractions can have more than one meaning. **Parts I and II of Probe 9.2: Rational-Number Concepts and Representations** (page 9-8 of the *Student Guide*) can help them see that rational numbers can have more than a part-of-the-whole meaning. A discussion of Part II can also help students distinguish among fraction terms (e.g., proper fractions and unit fractions) that may have little or no meaning for many. Note that a good many students will not know the difference between common fractions (which represent rational numbers and come in the form of $a/b$, where $a$ and $b$ are integers and $b \neq 0$) and the more general concept fractions (which can represent irrational numbers as well as rational ones). **Part III** (pages 9-8 and 9-9) is intended to help students reflect on the differences between common fractions and whole numbers, something which many may not have done before.

**Probe 9.3: Solving Fair-Sharing Problems Informally** (page 9-10 of the *Student Guide*) is intended to help readers see that children can invent their own informal solution strategies for solving fair-sharing problems and that such problems can serve as a basis for concretely introducing fraction concepts. Note that, because it is contrary to the personal experience of the vast majority of students, instructors may need to emphasize the point that rational numbers can be introduced informally (in the prefraction phase) with fair-sharing problems—a quotient meaning.

Unit 9•2: Common Fractions

Whereas Unit 9•1 focuses on the meanings,
learning, and teaching of rational-number concepts, Unit 9.2 focuses on meaningful learning and teaching of the verbal and written representations of rational-number concepts, namely fraction names and fraction notation. **Investigation 9.2:** Using Cuisenaire Rods to Develop Common Fraction Concepts (pages 9-16 and 9-17 of the Student Guide) can help adult students see how using manipulatives to solve challenging tasks can help children construct a deeper understanding of common fractions and master skills such as comparing two common fractions.

**SAMPLE LESSON PLANS**

**Project-Based Approach**

One possible project would be to have adult students individually interview children and assess their knowledge of common fractions. Using **Probe 9.A:** Gauging Children’s Informal and Formal Knowledge of Common Fractions (pages 240 to 242 of this guide), students working in small groups of about four could test children at, say, first, third, and fifth grade. Each group could compile and share its results. One focus of a class discussion could be commonalities and differences among the groups’ results. Another focus could be what a teacher might do to help children learn the fraction concepts and skills tested. Such a project would create a need for exploring the contents of chapter 9. Collecting and analyzing genuine data would create a need for exploring the contents of chapter 13. Moreover, the project would provide experience interviewing children in a fairly structured way.

An alternative project is to have small groups of students evaluate the fraction instruction of an elementary textbook or curriculum. See, for example, Suggested Activity 4 on page 250 of this guide. Note that either this project or that described above models the investigative approach.

**Single-Activity Approach**

By focusing on **Investigation 9.1:** Understanding the Conceptual Basis for Fractions (pages 9-4 and 9-5 of the Student Guide), a class could explore the part-of-the-whole and quotient meanings of common fractions, the central importance of equal partitioning in constructing a fraction concept, and how solving fair-sharing problems concretely could provide a basis for fraction instruction. Thoroughly exploring this investigation could take most, if not all, of one class period and models one way the investigative approach can be used to teach common fractions.

**Multiple-Activities Approach**

To provide a relatively comprehensive and systematic coverage of chapter 9 content, an instructor might choose to do a portion of each of the reader inquiries in the following sequence:

1. **Questions 1 and 2d to 2f of Probe 9.1:** Discrete Versus Continuous Quantities (page 9-3 of the Student Guide) can help students recognize that while discrete quantities consist of separate whole items with no in-betweens and, thus, can be counted, continuous quantities can be subdivided into parts and, thus, have in-betweens that can be quantified only through measurement. Note that Item 2f can help underscore the point that how a quantity is classified depends on how it is conceptualized (i.e., discrete and continuous quantities are concepts that humans construct and a classification system that we impose on the world).

2. **Problem 1 in Part I and Part III of Investigation 9.1:** Understanding the Conceptual Basis for Fractions (pages 9-4 and 9-5 of the Student Guide) can serve as a basis for discussing (a) the conceptual basis for fractions, namely equal partitioning, and (b) the key role fair-sharing problems can play in fraction instruction.

3. **Part I of Probe 9.2:** Rational-Number Concepts and Representations (page 9-8 of the Student Guide) can serve to introduce students to four rational-number meanings and, perhaps, expand their view of what a fraction can represent. A discussion of the questions in Part III (pages 9-8 and 9-9) can help them better understand why a fraction concept is conceptually difficult for students.

4. **Questions 1 and 2 in Probe 9.3:** Solving Fair-Sharing Problems Informally (page 9-10 of the Student Guide) can serve as a basis for discussing the strengths and weakness of children’s informal fair-sharing strategies. An instructor may wish to note that by sharing their solution methods and solutions, children can gain a valuable lesson in equivalent fractions: Solution Method A (one that the ancient Egyptians also used) leads to an answer of \(1 + \frac{1}{2} + \frac{1}{10} = 1 \frac{5}{10}\); Solution Method B, \(1 \frac{3}{5}\); and Solution Method C, \(\frac{8}{5}\)— answers that are equivalent. It may also be worth noting that primary-level children who use Solu-
tion Method A may recognize that each share is more than 1 \( \frac{1}{2} \) but may be unable to specify exactly how much, or they may incorrectly conclude that each share is 1 + \( \frac{1}{2} \) + \( \frac{1}{5} \) — that is, view the smallest piece of each share as a fraction (\( \frac{1}{5} \)) of the half, rather than as a fraction (\( \frac{1}{10} \)) of a whole. Note that this error can lead to a discussion about the importance of defining the whole when solving fraction problems.

5. Doing selected portions of Investigation 9.2: Using Cuisenaire Rods to Develop Common Fraction Concepts (pages 9-16 and 9-17 of the Student Guide) can illustrate how explorations with Cuisenaire rods can help children construct a number of important fraction-related ideas (see page 258 of this guide for details). It might be helpful to address each question covered as a class and then discuss the pedagogical significance of each (see the Questions for Reflection on page 9-17 of the Student Guide).

6. Activities II and III of Investigation 9.3: Informally Comparing Common Fractions (pages 9-20 to 9-22 of the Student Guide) can serve to introduce students to two more useful manipulatives for teaching fraction skills and concepts, namely, Fraction Circles and Fraction Tiles. Instructors may wish to have their students complete this investigation and Investigation 9.2 with manipulatives they think are particularly useful (in addition to or instead of those illustrated). Instructors who chose to do Activity IV (pages 9-22 and 9-23 on the "rectangular cake-cutting analogy" may need to emphasize that Figure A does not model this analogy. The first step in this model is illustrated by Figure B. The second step would be cutting up both cakes in this figure into equal-size pieces.

SAMPLE HOMEWORK ASSIGNMENTS

Read: Chapter 9 of the Student Guide.

Study Group:

- Questions to Check Understanding: 2, 3, 5, 10, 11a, and 12 (pages 251 and 252).
- Writing or Journal Assignments: 2 and 7 (pages 253 and 254).
- Problem: Burned by a Bad Investment (page 255).
- Bonus Problem: Fractions in Other Bases (pages 254 and 255).

Individual Journals: Writing or Journal Assignment 1 (page 253).

FOR FURTHER EXPLORATION

ADDITIONAL READER INQUIRIES

Probe 9.A (pages 240 to 242)

Gauging Children's Informal and Formal Knowledge of Common Fractions can help students (a) better appreciate children's informal knowledge, (b) recognize the importance of solving fair-sharing problems and an equal-partitioning concept in constructing an understanding of common fraction, and (c) acquaint them with performance-assessment tasks and error analyses.

Investigation 9.A (page 243)

Representing a Common Fraction with a Point on a Number Line can serve to further expose students to a quotient meaning of rational numbers. It can also serve to illustrate yet another way of representing a fraction: using number line.

Probe 9.B (page 244)

Indirect-Correspondence Models can help students explicitly compare and contrast such models with direct-correspondence models. This can help them see why equating the former with fraction symbols is relatively difficult for children to understand and represents a relatively deep understanding of common fractions.

Investigation 9.B (page 245)

The Sliced-Square Analogy for the Fraction-Renaming Algorithms illustrates how guided discovery-learning can help children rediscover the fraction-renaming algorithms.

Probe 9.C (pages 246 and 247)

Defining the Whole can help underscore the important heuristic of identifying the whole when solving fraction problems. Awareness of this heuristic should help teachers identify and avoid fraction problems in which the whole is ambiguously defined. (Text continued on page 248.)
Probe 9.A: Gauging Children’s Informal and Formal Knowledge of Common Fractions

The aim of Parts I to IV is to help familiarize you with children’s informal knowledge of fractions, particularly their knowledge of equal partitioning. Pose the tasks described in Parts I, II, and III to some kindergartners and first graders, or even older children. Pose the questions listed in Part IV to several third-, fourth-, and fifth-graders. Note whether a child is successful or not and what strategy or strategies are used to answer each item. Whether or not you actually interview children, answer the questions in Part IV yourself. What strategy or strategies did you use? Compare your efforts with the strategies used by others in your group. Then consider the Questions for Reflection at the end of the probe.

The aim of Parts V and VI is to familiarize you with some tasks for assessing children’s understanding of common fractions and to provide practice analyzing some common errors. Make enough copies of Part V (pages 241 and 242) so that you can interview several children at different ability levels from grades 1 to 5. Whether or not you actually interview children, complete Part VI on page 242. Discuss your findings or analyses with your group or class.

Part I: Equal-Partitioning (Fair-Sharing) Tasks Involving Discrete Quantities

Tasks 1 and 2: Ask a child to share 6 objects fairly between two people (Task 1) and 12 objects among three people (Task 2). If a child successfully uses a divvy-up strategy, ask if each share has the same number. If the child begins to count, ask: “Can you tell me without counting?” Note whether the child can respond without counting.

Part II: Equal-Partitioning (Fair-Sharing) Tasks Involving Continuous Quantities

Task 3: Show a child a clay hot dog. Tell the child two friends want to share the hot dog fairly. Ask the child to show you how the hot dog can be shared so that both friends get a fair share. Score the child as successful if the pieces are basically equal in size and none of the hot dog (clay) is left over. Note if the child had to use a trial-and-error procedure or could plan the cut in advance. Was this task easier or more difficult than Task 1?

Task 4: If a child is successful on Task 3, present another clay hot dog and ask how it could be shared fairly among three friends. Score Task 4 in the same way as Task 3. Compare the relative ease of this task with Tasks 2 and 3.

Task 5: Ask how another clay hot dog could be shared fairly among 4 friends.

Tasks 6 and 7: As a warm-up task, ask several children to fold a sheet of paper in half (into halves or two pieces). (a) Then ask several children to fold a sheet of paper into fifths. If they are primary age, ask them to fold the paper into five pieces of the same size (Task 6). (b) Ask the same children to fold another sheet of paper into thirds or three pieces (Task 7).

Tasks 8 and 9: (a) Ask children to divide a circle into halves (Task 8). (b) Ask them to divide another circle into thirds (Task 9).

Part III: Fraction-Of Tasks

Task 10: Place 6 objects (e.g., candies, blocks, or paper cookies) on a table. Ask the child to give you one-half. Are children more or less successful on this task than on Task 1, which did not use the term one-half?

Task 11: Ask a child to give you one-third of 12 objects. Are children more or less successful on this task than on Task 2, which did not use the term one-third?

Task 12: Ask a child to give you one-half of a clay hot dog. Are children more or less successful on this task than on Task 3?

Task 13: Ask a child to give you one-third of a clay hot dog. Are children more or less successful on this task than on Task 4?

Part IV: Ordering-Fractions Task

For each case below, can you tell which child got more?

a. Aggie won $\frac{4}{7}$ of the prizes; Beryl, $\frac{3}{7}$. Which is more: $\frac{4}{7}$ or $\frac{3}{7}$?
b. Crystal learned \( \frac{2}{5} \) of the spelling words; Dana, \( \frac{2}{5} \). Which is more: \( \frac{2}{5} \) or \( \frac{2}{5} \)?

c. Ervin’s hamburger weighed \( \frac{1}{4} \) pounds; Florence’s, \( \frac{1}{3} \) pounds. Which is more: \( \frac{1}{4} \) or \( \frac{1}{3} \)?

d. Gordon’s bag of candy weighed \( \frac{2}{3} \) pounds; Hacib’s, \( \frac{1}{2} \) pounds. Which is more: \( \frac{2}{3} \) or \( \frac{1}{2} \)?

e. Ivan read \( \frac{4}{7} \) of a computer manual; Jocelyn, \( \frac{2}{5} \) of the manual. Which is more: \( \frac{4}{7} \) or \( \frac{2}{5} \)?

f. Kim collected garbage that weighed \( \frac{9}{10} \) pounds; Lauren’s effort weighed \( \frac{11}{12} \) pounds. Which is more: \( \frac{9}{10} \) or \( \frac{11}{12} \)?

Part V: Formal Tasks

1. Color in one-fourth (\( \frac{1}{4} \)) of the circles:

2. Write a common fraction to show what part of the pie was eaten (the darker portion).

3. Write a common fraction to show what part of the pizza was eaten (the darker portion).

4. Which of these pictures shows one-half (\( \frac{1}{2} \))? Circle all correct answers.

   a.  
   b.  
   c.  
   d.  
   e.  
   f.  
   g.  
   h.  

5. What would happen to the size of the common fraction \( \frac{\bigtriangleup}{\bigtriangledown} \) in each situation below?

   Write L if the fraction would get larger; S, if it would get smaller; and N, if it would not change in size. Write A if anything is possible—that is, the fraction might increase, decrease, or remain the same in size.

   a. If the top term covered by the triangle got smaller and the bottom term covered by the box remained the same?

   b. If the term covered by the triangle remained the same and the term covered by the box got larger?

   c. If the terms covered by the triangle and the box both got smaller?

   d. If the terms covered by the triangle and the box both got larger?

   e. If the term covered by the triangle got larger and the term covered by the box got smaller?

   f. If the term covered by the triangle got smaller and the term covered by the box got larger?

6. The fraction \( \frac{1}{4} \) is equal to which of the following common fractions? Circle any correct answer.

   a. \( \frac{2}{8} \)  
   b. \( \frac{2}{5} \)  
   c. \( \frac{2}{4} \)  
   d. \( \frac{3}{12} \)  
   e. none of the above  
   f. all of the above

7. Over the weekend, Erna ate one-half of Grandma Brown’s one-layer strawberry cake. During this same time, Edie ate one-third of Grandma Brown’s three-layer chocolate cake. Can you tell which child ate more?
Probe 9.A continued

8. For each item below, circle the larger common fraction.
   a. \(\frac{1}{4}\) or \(\frac{1}{3}\)
   b. \(\frac{2}{3}\) or \(\frac{3}{8}\)
   c. \(\frac{4}{7}\) or \(\frac{2}{5}\)
   d. \(\frac{5}{8}\) or \(\frac{4}{6}\)
   e. \(\frac{8}{11}\) or \(\frac{5}{7}\)

Part VI: Analyzing Common Fraction Errors

Analyze each of the following errors. Generate a plausible hypothesis to explain the basis of each. That is, make an educated guess about how the child’s understanding of common fractions is incomplete or inaccurate.

1. In response to Question 1 in Part V above, Ramos asked, “One-fourth of a circle?” When the original question was repeated, Ramos appeared confused and colored in \(\frac{1}{4}\) of the first circle.

2. In response to Question 2 of Part V above, Audrey wrote \(\frac{1}{5}\).

3. In response to Question 3 of Part V, Rashad wrote \(\frac{2}{3}\).

4. For Question 4 of Part V, Edith indicated that figure a, f, h, l, and o (as well as b, c, e, i, j, and p) show \(\frac{1}{2}\). She did not circle d, g, k, m, or n.

5. In response to Question 6 of Part V, Naomi circled \(\frac{2}{3}\). In response to follow-up questions, she chose \(\frac{2}{3}\) as the equivalent for \(\frac{1}{2}\) and \(\frac{2}{6}\) as the equivalent for \(\frac{1}{3}\).

6. a. On Question 8 of Part V, Sergei chose (a) \(\frac{1}{4}\), (b) \(\frac{1}{9}\), (c) \(\frac{1}{7}\), (d) \(\frac{5}{8}\), and (e) \(\frac{8}{11}\) as the larger.

   b. Payne did not answer Question 8a and chose as the larger (b) \(\frac{3}{8}\), (c) \(\frac{4}{7}\), (d) \(\frac{5}{8}\), and (e) \(\frac{8}{11}\).

   c. Eleanor chose (a) \(\frac{1}{7}\), (b) \(\frac{1}{9}\), (c) \(\frac{1}{7}\), (d) \(\frac{5}{6}\), and (e) \(\frac{5}{7}\).

   d. Dori chose (a) \(\frac{1}{4}\), (b) \(\frac{1}{9}\), (c) circled both \(\frac{1}{7}\) and \(\frac{2}{7}\), (d) \(\frac{5}{7}\), and (e) \(\frac{8}{11}\).

Questions for Reflection

1. Try folding a sheet of paper into (a) halves, (b) thirds, and (c) fifths yourself. Specify a procedure for accomplishing each task.

2. Cut out a circle. Try folding it into (a) halves and (b) thirds.

3. (a) In Part II, did successful children use the same procedure for dividing a continuous quantity into three parts as they did for doing so into two parts? (b) What errors did children commonly make when trying to divide a continuous quantity into thirds? (c) Did children (or you) use the same procedure to subdivide a sheet of a paper into five parts as it took to do so into three parts or not? Was the procedure identical to that for subdividing a sheet of paper into two parts?

4. (a) For Part II, did children use the same procedure to subdivide different quantities (a clay hot dog, a piece of paper, and a circle) into two (or three) parts? (b) Did you use the same procedure to subdivide a circle as you did to subdivide a rectangular piece of paper?

5. What do you conclude about the relative ease of equally partitioning discrete and continuous quantities? Are they equally difficult or is one more difficult than the other. Why?

6. Analyze the following responses to Task 12 (cutting a clay hot dog in half). In what way does each child’s understanding of one-half appear incomplete?

   a. Robin started at one end of the hot dog and repeatedly cut it.
   b. Stacey rolls the clay hot dog into a ball.
   c. Tripoli cut the hot dog into unequal pieces.
   d. René cut the hot dog into two unequal pieces and then lopped off a piece to make each half the same size.
   e. Uwi cut the hot dog into two unequal pieces, lopped off a piece to make each half the same size, halved the loped-off portion; and attached a piece to each half.
Investigation 9.A: Representing a Common Fraction with a Point on a Number Line

This investigation illustrates how a number line can be used to represent a quotient meaning of rational number. To see what is involved, work through the investigation yourself.

Part I: Using a Number Line to Raise Questions About Fractions. For the number line below, the vertical hash marks represent whole numbers. Starting at the origin and jumping three to the right would put you at +3; where would starting at the origin and jumping half this far put you (Page, 1964)?

Part II: Using a Number Line to Connect a Quotient Meaning to Fraction Symbols. The following graphing activity can help children see that symbolic division can be represented as a fraction, which in turn, can be represented on a number line. For example, it can help convince children that $3 ÷ 4 = \frac{3}{4}$ and that both are equivalent to the number-line representation:

1. With children, begin the activity with some examples that involve whole-number answers. Probe 5.4 (page 523 of the Student Guide) illustrated how a division expression such as $8 ÷ 4 = 2$ can be modeled on a number line: (a) use a line segment 0 to 8 to represent the dividend 8, (b) divide the segment into four equal pieces to represent the divisor 4, and (c) note that each piece (the quotient) is two. What meaning of division is represented by this model?

   (a) Paper strip is cut to match the length of the line segment 0 to 8:

   (b) Paper strip is folded into 4 equal-sized pieces:

2. (a) Now consider the expression $3 ÷ 4$. Based on the example above, illustrate below how you could use a paper strip and a number line to represent the dividend, the divisor, and the quotient. (b) Do the same for the expression $2 ÷ 3$.

Part III: Using a Number-Line Representation to Order Fractions. Children can create their own number-line representation of the fourths family, for instance, by (a) dividing a line segment into 4 equal parts, (b) labeling the ends of the line segment 0 and 1, (c) labeling the intermediate divisions $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$. These three steps are illustrated in the figures below. By repeating steps a to c, other fraction families like thirds, fifths, and sixths can be entered on the same number line. Create your own number line to compare thirds, fourths, fifths, and sixths.

   (a) Fold a paper strip into 4 parts.

   (b) Define the length of the strip as the whole.

   (c) Note fourths.
Some fraction models correspond directly with a common fraction. For example, the model \( \bullet \bullet \circ \circ \circ \) corresponds directly to \( \frac{3}{4} \) because two of the four circles are dark. This model can also be represented by \( \frac{1}{2} \), but it does not correspond directly to this common fraction. Children who understand the relationship between fractions and indirect-correspondence models have made an important step in constructing a deep understanding of common fractions.

1. Color in \( \frac{1}{4} \) of Figures A, B, and C below.

![Figure A](image1)
![Figure B](image2)
![Figure C](image3)

2. Figures A, B, and C above are area models of \( \frac{1}{4} \). (Models A and B could be drawn on graph paper; Model C could be drawn on dot paper or constructed on a geoboard.) Model A is a direct correspondence model of \( \frac{1}{4} \) because the number of colored portions (one) corresponds exactly to the numerator (1) and the total number of portions (four) corresponds exactly to the denominator (4). Model B is an indirect-correspondence model because the number of colored portions (four) does not correspond exactly to the numerator (1) and the total number of portions (sixteen) does not correspond exactly to the denominator (4). (Note, though, that Model B is a direct correspondence model of \( \frac{4}{16} \).) Likewise, Model C is an indirect-correspondence model of \( \frac{1}{4} \). For each medium indicated below, illustrate how you could create an indirect-correspondence model of \( \frac{1}{4} \).
   a. A length model using Cuisenaire rods:
   b. A weight model using blocks and a balance beam:
   c. A discrete-quantity model using colored counters:
   d. A discrete-quantity model using an egg carton:

Questions for Reflection

1. Examine the models of \( \frac{1}{4} \) in Figure 9.8 (page 9-18 in the Student Guide). Are these direct- or indirect-correspondence models?

2. (a) Illustrate various ways how Figure B could be used to represent one-fourth. (b) Illustrate various ways Figure C could be used to do the same. (c) How might this variety of examples serve to extend children’s understanding of one-fourth?

3. Is the model to the right an indirect-correspondence model of \( \frac{1}{3} \)? Why or why not?
Investigation 9.B: Sliced-Square Analogy for the Fraction-Renaming Algorithms

♦ Discovering the fraction-renaming algorithms ♦ 4-8 ♦ Class as individuals

This investigation illustrates a structured discovery-learning lesson for helping children rediscover the fraction-renaming algorithms. Try the activity yourself to see what is involved.

To start, shade in \( \frac{3}{4} \) of Square A. Complete the table at the bottom of the page by following steps a to f described below.

a. Use a horizontal line to subdivide Square A further into equal parts. What is the number of shaded parts now? What is the total number of parts now? Record these data under Step a in the table below.

b. Shade in \( \frac{3}{4} \) of Square B. Then use two horizontal lines to further subdivide Square B into equal parts. What is the number of shaded parts now? What is the total number of parts now? Record the data under Step b in the table below.

c. Shade in \( \frac{3}{4} \) of Square C. Then use three horizontal lines to further subdivide Square C into equal parts. What is the number of shaded parts now? What is the total number of parts now? Record these data under Step c in the table below.

d. Shade in \( \frac{3}{4} \) of Square D. Then use four horizontal lines to further subdivide Square D into equal parts. What is the number of shaded parts now? What is the total number of parts now? Record these data under d in the table below.

e. Imagine shading in \( \frac{3}{4} \) of square as you did to begin Steps a to d. Now imagine using five horizontal lines to further subdivide the square into equal parts. How many shaded parts are there? How many total parts are there? Record your answers under Step e in the table below.

f. Imagine repeating Step e but this time using six horizontal lines to further subdivide the square into equal parts. Predict the number of shaded parts and the total number of parts and record these predictions under Step f in the table below.

1. Examine the table below. (a) Is there a shortcut for renaming \( \frac{3}{4} \) the equivalent fraction produced by Step a? (b) Step b? (c) Each of the other Steps c to f? (d) What rule could be used to rename \( \frac{3}{4} \) to any equivalent fraction?

2. What rule could be used to reduce any equivalent fraction of \( \frac{3}{4} \) to simplest terms, that is, to \( \frac{3}{4} \)?

**Teaching Tip.** Note that the same activity could be done with paper folding or with a square drawn on graph paper. For example, ask children to fold a paper strip into fourths. Have them color one-fourth. Then challenge them to fold other strips into thirds, fifths, and so on and find a model of a fraction that has the same length as one-fourth. Neither one- or two-thirds have the same length as \( \frac{1}{4} \), for example, but \( \frac{2}{8} \) does.

<table>
<thead>
<tr>
<th>Step a</th>
<th>Step b</th>
<th>Step c</th>
<th>Step d</th>
<th>Step e</th>
<th>Step f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square A before</td>
<td>Square A after</td>
<td>Step b (Square B)</td>
<td>Step c (Square C)</td>
<td>Step d (Square D)</td>
<td>Step e</td>
</tr>
<tr>
<td>Number of shaded parts</td>
<td></td>
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</tr>
<tr>
<td>Total number of parts</td>
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</tr>
</tbody>
</table>
The aim of this probe is to help you reflect on the importance of explicitly defining the whole in a fraction problem.\(^1\)

**Part I: The Two-Pizza Questions**

Consider the following situation: May and June ordered two small-sized pizzas from Pizza Pig-Out. Each pizza was divided into four equal pieces. The shaded area in the diagram to the right represents the pieces that were eaten.

Answer the Questions A to D below on your own. Express all answers in fraction form. Then answer **Questions to Consider** that follow. Discuss your answers to Part I with your group or class.

**Question A:** How many pizzas did May and June eat?

**Question B:** What fraction of the pizza pieces did May and June eat?

**Question C:** What portion of the pizzas in May and June’s order were completely eaten?

**Question D:** How much pizza did May and June eat?

**Questions to Consider for Part I**

1. Are Questions A, B, C, and D equivalent? That is, do they have the same answer?
2. The question *What fraction of a pizza did May and June eat?* is equivalent to which question(s)—A, B, C, and/or D?
3. Were any of the questions unclear to you?
4. (a) Question A above implies what whole? (b) Question B implies what whole? (c) Question C implies what whole? (d) Question D implies what whole?

**Part II: The Case of the Confused Whole**

Darrin got his seatwork assignment back (see the figure on the next page) and was most pleased with the feedback.

LeMar, on the other hand, was surprised that his answer of \(\frac{14}{24}\) for Question 3 was marked wrong. He asked Darrin what his answer was and how it was graded.

Darrin responded proudly, "Why 14 twelfths, of course, *the* correct answer."

Unconvinced, LeMar replied, "I don’t think so. I think it’s 14 twenty-fourths."

"Well 14 twenty-fourths and 14 twelfths are the same thing. No, wait, they aren’t," Darrin quickly added. "Look, this \(\frac{14}{12}\) has to be the correct answer. A pizza has 12 pieces and 14 pieces were eaten, so 14 twelfths of a pizza were eaten. See, Miss Brill has marked it correct."

A skeptical LeMar noted, "But the question says, 'What fraction of the *two* pizzas was eaten?' There are 24 pieces in the two pizzas, 14 were eaten, and so 14 twenty-fourths of the pieces were eaten."
Pizza Problems

Pizza Pig-Out cuts their large pizzas into 12 pieces. Mr. Plague’s class ordered two large pizzas. Seven children from Mr. Plague’s class ate 2 pieces of pizza each.

1. Choose a manipulative and make a model of this situation.

2. Make a drawing of your model. Shade in the amount eaten.

3. What fraction of the two pizzas were eaten? $\frac{14}{12} \; 0.\bar{8}$

4. What fraction of the two pizzas was each student’s share? $\frac{2}{12} \; 0.\bar{8}$

5. How much pizza was left uneaten? $\frac{10}{12} \; 0.\bar{8}$

Miss Brill, who earlier would simply have put the kibosh on the argument that broke out between Darrin and LeMar, instead asked each to state his case for the class. Moreover, she paid careful attention to LeMar’s explanation for his “incorrect” answer. Indeed, when the class concluded that the correct answer to Question 3 was $\frac{14}{24}$, not $\frac{14}{12}$, Miss Brill graciously admitted she had just learned something.

Then as a team, Miss Brill and the class set about examining Questions 4 and 5. After a brief debate about Question 4, they turned their attention to Question 5. Darrin felt that his answer of $\frac{10}{12}$ was correct, because there were 12 pieces in a pizza, 2 had been eaten, leaving 10 pieces or $\frac{10}{12}$ of the partially eaten pizza. LeMar noted that there were 24 pieces of pizza altogether and 14 pieces were eaten. That leaves 10 pieces of pizza or $\frac{10}{24}$ of the total pizza pieces. The class could not decide who was correct.

Jane was the first to see the difficulty and commented, "Maybe both 10 twelfths and 10 twenty-fourths are correct. It just depends on how you interpret the question. You could view the whole as the partially eaten pizza or as the set of two pizzas."

Miss Brill, who earlier in the year would have tried to hide her error, commented: "My question does seem unclear. I think we can all learn something from this." The incident served to underscore the importance of using language carefully and provided a basis for discussing why it is important to identify the whole when solving fraction problems.

Questions to Consider for Part II

1. How could Question 3 be rewritten so that the correct answer is $\frac{14}{12}$?

2. What is the correct answer to Question 4?

3. How could Question 5 be rewritten so that it more clearly implied a whole of 24 pieces?
QUESTIONS TO CONSIDER

1. The ancient Egyptians had special symbols for certain fractions: \( \text{\textbullet} \) for \( \frac{1}{2} \), \( \text{\textbullet}\text{\textbullet} \) for \( \frac{1}{4} \), and \( \text{\textbullet}\text{\textbullet}\text{\textbullet} \) for \( \frac{1}{7} \) (Bunt et al., 1976). Otherwise, unit fractions were noted by a special symbol (e.g., \( \frac{1}{5} \) was recorded as \( \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \)). Nonunit fractions were represented by using combinations of unit fractions. Answering the following questions should help you understand this system better.

   a. Write \( \frac{1}{12} \) in Egyptian hieroglyphics. What part of a fraction does the special symbol above the numeral represent? What part of a fraction does the symbol for the counting number represent?

   b. In time, the ancient Egyptians abandoned the unique symbol for \( \frac{3}{4} \), perhaps because it was too easily confused for \( \frac{3}{1} \). In its place, the ancient Egyptians used a combination of two symbols. What two symbols would be a logical choice for representing \( \frac{3}{4} \)? Hint: It cannot be \( \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \) because the ancient Egyptians used only unit fractions.

   c. Asked to illustrate how the ancient Egyptians would have represented \( \frac{7}{12} \), one group of students suggested \( \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \). This is a logical answer. However, the ancient Egyptians discovered a more compact way of combining unit fractions to represent nonunit. (Hint: The method parallels the informal Solution A illustrated in Figure 9.3 on page 9-10 of the Student Guide).

   d. How do you suppose they may have represented the solution to the following situation? El brought home 3 loaves of bread to share with his wife and 3 children. If shared fairly, how many loaves of bread did each of the five members of El’s family get? Hint: The ancient Egyptians’ solution was intuitive.

   e. How do you suppose they represented the solution to the following problem? Flog bought 3 loaves of bread to share with his squad. If shared fairly among 10 members of the squad, how many loaves of bread would each soldier get?

   f. Illustrate with Egyptian hieroglyphics how the nonunit fractions \( \frac{7}{13} \), \( \frac{7}{8} \), \( \frac{3}{5} \), \( \frac{17}{20} \), and \( \frac{5}{24} \) could be represented with unit fractions. Can you devise a general procedure for representing any nonunit fraction as the sum of two or more unit fractions?

   g. Problems such as Questions d and e above can raise issues that naturally lead to instruction about key fractions concepts. Consider the case of Alison, who was just about to turn 9 and who had been introduced to fractions in school. After hearing Question d above, Alison immediately suggested cutting the loaves in half but then gave an incomprehensible answer. Alexi chimed in with, “I know how to do it.” Encouraged to use a drawing, he divided the three loaves into halves and divided the remaining half into five pieces. He concluded each member of the family would get one-half and one-something.

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At this point, the teacher noted that one piece of five was correct for half a loaf of bread and asked what it would be for the whole loaf? Alison asked: "How many pieces here [in the unsegmented half of the third loaf]?

In return, the teacher asked: "How many here [in the segmented half]?"

Alison: "Oh, five. So there are five."
The teacher then asked: "So what part of a whole is this piece?"

Alison then concluded that the partial share was *one-tenth.*

What issue was raised by solving the problem? That is, to determine what "one-something" was, what fraction concept needed to be extended?

2. Why it might be useful to introduce elementary-school children to the fraction system of the ancient Egyptians?

3. As noted in chapter 5 of the *Student Guide,* division can have both divvy-up and measure-out interpretations. In chapter 9 of the *Student Guide,* it was noted that fraction notation such as \( \frac{1}{3} \) can be given a divvy-up meaning: "one shared with three." How could \( \frac{1}{3} \) be read so that a measure-out interpretation applies?

4. With a part-of-a-whole interpretation, \( \frac{8}{4}, \frac{5}{4}, \frac{2}{4}, \frac{1}{4} \), and \( \frac{1}{4} \) are modeled with either continuous or discrete quantities. With a divvy-up (quotient) interpretation, \( \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \) and \( \frac{1}{4} \) are modeled with a continuous quantity. The fraction representing the quotient indicates the size of each person's share.

<table>
<thead>
<tr>
<th>Number of Pizzas</th>
<th>Divided Fairly With</th>
<th>Size of Each Person's Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>( \frac{4}{2} )</td>
<td>( \frac{2}{1} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{4}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

(a) However, can a fair-sharing (quotient) interpretation of \( \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \) and \( \frac{1}{4} \) all be modeled with discrete quantities? Why or why not? Is expressing the quotient (i.e., the size of a single share) in fractional terms practical for each case? (b) What are the instructional implications of your conclusion?

5. The following software lesson is drawn from the *Instructor’s Guide to the CCA Basic Skills Mathematics Curriculum* published in 1990 by the University of Illinois Computer-based Education Research Laboratory. The lesson begins by introducing fraction equivalents equal to one. The student is shown a pizza precut into, for instance three equal slices and one uncut pizza \( \frac{1}{3} = 1 \). The student is asked if the pizzas can be shared fairly between two people—without further cutting.

Next a student is asked if the pizza shown below can be shared fairly between two people, without leaving any leftover.

If a student gives each person 3 pieces (and leaves a whole pizza and one piece leftover), the computer complains: "That's not fair. You have pizza leftover!"

a. What solution will the computer accept?

b. What concept does this lesson address?

6. A can of Classic Coke contains 9 teaspoons of sugar. Diet Coke uses aspartine, a sugar substitute that is 200 times sweeter than sugar. Assuming Classic Coke and Diet Coke have the same volume and are equally sweet, how many teaspoons of aspartine are in Diet Coke? (This explains why a can of Diet Coke floats in water while a can of Classic Coke goes unders.) Antoinette answered 1800. Is she right or wrong? Briefly explain.

7. Consider the following word problem:

- **Plain Pizza** (6-8). Cosmo planned to have 10 people at his birthday bash some of whom were vegetarians. He wanted to serve each person one-half of a mediumsized pizza. So he had Pizza Pig Out prepare two plain pizzas and three with pepperoni and sausage. What fraction of the servings were suitable for vegetarians?

   (a) However, can a fair-sharing (quotient) interpretation of \( \frac{3}{4}, \frac{5}{4}, \frac{2}{4}, \frac{1}{4} \) all be modeled with discrete quantities? Why or why not? Is expressing the quotient (i.e., the size of a single share) in fractional terms practical for each case? (b) What are the instructional implications of your conclusion?

(b) What is the whole? (c) Would it be considered a continuous or a discrete quantity? (c) What is the answer to the problem?

8. Consider the following fraction problem:

- **The Partisan Pizza** (6-8). Stanley was going to have 6 guys over to watch the Super Bowl. Not sure how to divide a circular
region into sevenths, Stanley divided the pizza he made into halves. He then divided one half into three servings (for himself and the two guys who liked his team). Next, he divided the other half into four servings (for the guys cheering for the other team). Finally, he put pepperoni on two of the servings because only he and Philip like pepperoni. What fraction of the servings had pepperoni?

(a) The problem above involves what kind of quantity? (b) What is the whole? (c) Is the question answerable? If so, what is it? If not, why not?

**SUGGESTED ACTIVITIES**

1. The ancient Egyptian symbols for fractions can serve a variety of instructional purposes. Question 1 in Questions to Consider can be tied to Social Studies lessons on ancient cultures. Questions 1d and 1e can serve as challenging problems for intermediate- and even primary-level students. Devise your own problems that parallel 1d and 1e.

2. Collect a number of egg cartons. Cut some up so that you have cartons of various sizes (cartons for 2 eggs, 3 eggs, 4 eggs, and so forth). (a) Consider how you can use these cartons and plastic eggs to illustrate the point that a fraction by itself expresses only a relative value. (b) For the previous question, Megan devised the following demonstration (the dark circles indicate an egg, the light circles indicate an empty depression). Evaluate Megan’s effort. Does it adequately address the issue? Why or why not?

   ![Egg Carton Diagram](image)

3. Devise a game board to play the (a) *Equivalent-Fraction Version*, (b) the *Smaller-Fraction Version*, and (c) the *Larger-Fraction Version* of *Fraction Trip* (Activity V in Investigation 9.3 on page 9-23 of the *Student Guide*).

4. Use the *Checklist for Evaluating Fraction Instruction* to analyze and evaluate a textbook, curriculum, or program. Write a report using a word processing tool. Share your assessment with others in your group or class.

**A Checklist for Evaluating Fraction Instruction**

<table>
<thead>
<tr>
<th>Name of the Material:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uses both discrete- and continuous-quantity models?</td>
</tr>
<tr>
<td>Explicitly teaches the heuristic of defining the whole?</td>
</tr>
<tr>
<td>Explicitly points out that fractions involve a part of so many equal parts?</td>
</tr>
<tr>
<td>Introduces fractions in a concrete context?</td>
</tr>
<tr>
<td>Introduces fractions in a fair-sharing context?</td>
</tr>
<tr>
<td>Instruction and practice are purposeful?</td>
</tr>
<tr>
<td>Uses a variety of models?</td>
</tr>
<tr>
<td>Explicitly links verbal fraction labels to concrete models?</td>
</tr>
<tr>
<td>Explicitly links concrete models and verbal labels to symbolic representations?</td>
</tr>
<tr>
<td>Employs reverse processing (see Figure 9.9 on page 9-19 of the <em>Student Guide</em>)?</td>
</tr>
<tr>
<td>Uses concrete models to introduce fraction comparisons?</td>
</tr>
<tr>
<td>Explicitly points out that fractions represent a relative relationship—that the whole must be identified to compare two fractions?</td>
</tr>
</tbody>
</table>

5. (a) Use Probe 9.A (pages 240 to 242 of this guide) as a basis for assessing the informal and/or formal fraction knowledge of several children at several different grade levels. Describe each child’s informal and formal strengths. What informal or formal knowledge appears incomplete? Analyze any errors made. What appears to be the basis for each error? (b) What are the instructional implications of your results? (c) Devise, implement, and evaluate a plan for remedial instruction. Indicate what investigations in the *Student Guide* or other worthwhile tasks were helpful. (d) Report your data and conclusions to your class.

6. (a) Use Probe 9.B (page 244) as a basis for assessing several third-, fifth-, and seventh-grader’s understanding of fractions. Does a child recognize indirect-correspondence models as a valid representation of a fraction? Can the child create indirect-correspondence models for fractions? (b) What are the instructional implications of your results? (c) Devise,
implement, and evaluate a plan for remedial instruction. (d) Report your data and conclusions to your class.

7. (a) Use an investigation, probe, or instructional idea from chapter 9 of the Student Guide as the basis for developing a lesson plan or a unit. (b) Try out your lesson or unit with a small group or class of elementary students. If possible, videotape the lesson(s). Evaluate your lessons, including the children’s disposition toward the lesson(s), what progress they made in understanding fractions, and how you might revise the instruction for future efforts. (c) Present your lesson or unit plan and your evaluation of it, including representative videotape segments, to your class.

8. Analyze and evaluate one of the instructional resources listed on page 9-27 of the Student Guide. Include in your evaluation what approach to mathematics instruction the author(s) had in mind and whether or not the suggested activities could be used in or adapted for the investigative approach.

9. (a) Choose a children’s book, such as those listed on pages 27 and 28 of the Student Guide and develop a lesson or unit about fractions around it. (b) Implement and evaluate the plan. (c) Share your experience and what you have learned with your class.

10. (a) Develop a list of educational software that involves fractions. Include a brief description of each item, including what aspects of fraction knowledge it involves, the appropriate grade level, and format. (b) Examine and evaluate the items. Include in your evaluation whether the program could serve as a worthwhile task and the basis for a lesson that embodies the investigative approach.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. An analog clock has an hour hand and a minute hand. A digital clock displays time using digits (e.g., 10:37). (a) An analog clock treats time as which, a continuous quantity or a discrete quantity? (b) A digital clock effectively treats time as which?

2. Draw a concept map that illustrates the relationship among (a) fair-sharing situations, (b) divvy-up meaning, (c) measure-out meaning, (d) operator meaning, (e) part-of-a-whole meaning, (f) quotient meaning, (g) ratio meaning, and (h) rational numbers. Include in your map the following examples: (i) Forty of the 100 students surveyed disagreed with the student council’s decision; (ii) For every four students surveyed who disagreed with the student council’s decision, six agreed; (iii) If Cherish and Portia earned $12 babysitting and they decided to split the earnings evenly, how much would each get? (iv) If Georges ate two pieces of a cake that was cut into eight equal pieces, what portion of the cake did he eat? (v) The probability of getting two heads in a row when flipping a coin is \( \frac{1}{4} \). (vi) The odds of getting two heads in a row when flipping a coin are \( \frac{1}{3} \), less likely than not getting two heads. (vii) If 12 agents were assigned equally to four states, how many agents were assigned to each state? (viii) If Maury put one of every $3 he earned in his saving account and had earned $12 over the weekend, how many dollars did he bank? (ix) Vuk had 24 tickets. If the airplane ride required three tickets, how many times could Vuk take this ride?

3. (a) Can every rational number be represented or named by a fraction? (b) The following symbol is a fraction: \( \frac{2}{5} \). Is this fraction a rational number? Why or why not? (c) What do your answers to (a) and (b) suggest about the relationships between rational numbers and fractions?

4. Which of the following represent a rational number: (a) \( 1 \frac{2}{7} \), (b) 0.25, (c) 4.5, (d) 3 ÷ 4, (e) 60 mph, (f) \[ \begin{array}{c|c}
0 & 1 \\
\hline
\end{array} \], (g) (2, 5)?

5. Demonstrate or illustrate how (a) Cuisenaire rods, (b) Fraction Circles, and (c) Fraction Tiles could be used to model the mixed number (complex fraction) \( 1 \frac{1}{2} \).

6. Researchers (Behr, Wachsmuth, Post, & Lesh, 1984) asked children to fill in a missing numerator or a missing denominator to make two fractions equal. Some examples are illustrated below. Children commonly responded to both the first and second expression below.
with an answer of 10. What is the basis of this systematic error?

\[
\frac{6}{4} = \square \quad \frac{3}{4} = \square
\]

7. Consider the question: What number is halfway between \(\frac{1}{2}\) and \(\frac{3}{4}\)? (a) Children commonly respond \(\frac{5}{8}\). Why might children give such an answer? (b) What fraction is halfway between \(\frac{1}{2}\) and \(\frac{3}{4}\)?

8. Martin and Marvin each bought their favorite candy bar at the store. Martin ate one-half of his candy bar; Marvin ate a third of his. Can you tell which boy ate more candy? Why or why not?

9. a. Yoshi reasoned that \(\frac{5}{8}\) is more than \(\frac{1}{2}\) because \(\frac{5}{8}\) is more than \(\frac{1}{2}\) and \(\frac{1}{2}\) is less than \(\frac{1}{2}\). Does this strategy entail intuitive, inductive, or deductive reasoning?

b. Tutu used \(\frac{1}{2}\) to informally reason that \(\frac{4}{6}\) was more than \(\frac{5}{8}\). Delineate a line of reasoning she could have used to determine her answer.

c. Gloria used a fraction as a benchmark to determine that \(\frac{5}{8}\) is more than \(\frac{6}{15}\). What fraction is used most often by children as a benchmark? Delineate Gloria’s line of reasoning based on this benchmark. That is, describe how she could have used this benchmark to arrive at her answer.

10. Illustrate how the rectangular cake-cutting analogy can be used (a) to determine whether or not \(\frac{1}{2}\) is equal to \(\frac{2}{6}\) and (b) to determine an equivalent fraction of \(\frac{2}{3}\).

11. (a) Illustrate how the rectangular-cake analogy could be used to compare the fractions \(\frac{2}{5}\) and \(\frac{4}{7}\). (b) Briefly explain how this analogy could be used to foster the discovery of the cross-products algorithm. (c) Finding a common denominator makes ordering fractions incomparably easier. How is the rectangular-cake analogy related to the common-denominator method?

12. Circle the letter of any of the following statements that—according to the Student Guide—is true. Change the underlined portion of any false statement to make it true.

a. Continuous quantities have no "in-between.

b. An example of a continuous quantity is the number of pupils in our class.

c. A fair-sharing word problem, such as How much candy will each boy get if four boys share 20 candies fairly? embodies a quotient meaning of rational numbers.

d. A fraction such as \(\frac{2}{3}\) represents an absolute value.

e. The next fraction after \(\frac{1}{5}\) is \(\frac{1}{6}\).

f. Prefraction instruction should begin with a part-of-a-whole meaning and verbally labeling concrete models.

g. Relating the fraction-bar notation to a part-of-a-whole or quotient meaning can help children see that \(\frac{1}{2} > \frac{1}{3}\),

h. For primary-age children, common fractions should be introduced with a variety of continuous quantities but not discrete quantities.

i. Use of mental imagery of concrete models of fraction should precede symbolically representing concrete models of them

j. As with whole numbers, estimation should play a significant part in fraction instruction.

13. a. Zeta Omega Omega fraternity took a bus trip to the zoo to visit some of its new pledges. They piled into two buses, filling 22 of the 24 seats on one bus and 21 of the 24 seats on the second bus. What fraction of the seats was filled?

b. The brothers of Zeta Omega Omega ordered six pizzas from Audi’s Day-old Recycled Pizzas for their Pledge Banquet. The pizzas were each divided into eight pieces. Having somewhat more sense than the fraternity brothers, the pledges ate only nine pieces. What fraction of a pizza did the pledges eat?
WRITING OR JOURNAL ASSIGNMENTS

1. Read Box 9.1 below. (a) The instruction on "halves" by the boy's mother involves what rational-number meaning? (b) The boy was quizzed on a problem that involved sharing an apple between two people. What rational-number meaning did the quiz embody and was it the same or different than that of his instruction? (c) Consider how the boy's mother might have helped him understand the "quiz" question better.

2. Efforts to relate division and fractions are often done in a superficial manner. Children are frequently taught to convert division expressions to fractions in a mechanical manner—without relating what they are doing to anything meaningful. For 7 ÷ 3, for example, they are informed that the quotient 2 r1 is equivalent to 2 1/3. Does it make sense to convert 2 r1 to a fraction when divvying up discrete quantities? When divvying up continuous quantities? Does it make sense to convert 2 r1 to a fraction when measuring out discrete quantities? When measuring out continuous quantities? Briefly justify your answers.

Box 9.1: A True Story About Fraction Instruction

When her son was in kindergarten, a high school mathematics teacher decided to work with him on fractions. She explained "halves" to him for about 30 minutes, even illustrating that two halves make up a whole, and a whole can be cut in two pieces called halves. After indicating he understood the lesson, the mother proposed a quiz. "Suppose your friend, Joey, came to play and you both wanted an apple to eat, but I only had one apple. What would I have to do?"

Son (S): "Go BUY Joey an apple."
Mother (M): "No, I don't have any money."
S: "Go to the bank and get some."
M: "No, the bank is closed."
S (cheerfully): "Then write a check."
M (starting to get peeved): "No, I'm not going to write a check, and I'm not going to the store!"
S: "Then give Joey a banana."
M: "No! No bananas! All I have is ONE apple! Now what is going to happen?"
S: "Sounds like you've got a fight on your hands."

3. (a) Miss Peach asked her second graders to write a fraction to indicate what part of the circle shown in Figure A below was shaded. Nora answered 1/4. Armond answered 2/4. Briefly explain the basis for each child's errors. (b) Asim was asked to write a fraction for Figures B and C below. He wrote 1/4 for Figure B and 1/2 for Figure C. What is Asim's error and how might a teacher help him overcome it?

4. The pamphlet Beginning to Learn Fractions (Research Information for Teachers Set, Item 13 written by Robert Hunting and published in 1989 by the NZCER and ACER) describes partitioning activities appropriate for primary children set in several real-world situations. The Farm Analogy, for example, is introduced in three phases.

In Phase 1 questions and activities focus on fostering or practicing a divvying-up procedure with sets of farm animals. Children might be told that a farmer has a load of pigs that he wants to put into two yards. Because the farmer wants the same number of pigs in each yard, the children are asked to distribute the pigs equally. The children are then encouraged to discuss various methods for accomplishing this objective and to explain why a systematic dealing method is best.

Phase 2 focuses on helping children to associate the conventional verbal labels for fractions with the results of a divvying-up strategy. For example, a teacher might ask: "Can you help me put half the chickens in one coop, and half in the other coop?" Children should discuss how this can be done. A teacher can ask the children to justify their procedure by asking, for example, "Are half the chickens in that coop?" Similar problems can be posed where farm animals are placed in three, four, five, and more yards, pens, barns, and so forth.

In Phase 3, a teacher might pose a problem such as: "The farmer has lost some of his ducks because the fence was broken. These are the
ducks left (indicate). These are one-half of all the ducks that were in the yard. How many ducks did the farmer have at the beginning? This problem can be repeated with different numbers of ducks (wholes) and later with thirds, fourths, fifths and so on. Written fraction notation is also introduced during this phase.

Evaluate the Farm Analogy. What step(s) in the four-step developmental instruction sequence do(es) each phase of the analogy represent? What specific step does the example in the description of Phase 2 illustrate? What type of problem is the example in the description of Phase 3?

5. Miss Brill thought it would be a good idea to have her class use Cuisenaire rods to solve some fraction problems concretely. She posed the following question: If the brown (8 cm) rod is \( \frac{1}{2} \), what rod represents a \( \frac{1}{8} \)? Many of the children seemed confused. Why?

6. You ask your student teacher Priscilla to make up some word problems on fractions. Priscilla comes up with following problem: George cut four candy bars into three pieces each. He ate \( \frac{1}{3} \) pieces. What fraction did he eat? Briefly evaluate Priscilla’s effort.

7. Mr. Snyder asked his class to represent the relationships among real numbers, rational numbers, irrational numbers, and fractions. Which diagram in Figure 9.1 below, if any, is correct? Justify your answer.

8. While substituting in a second-grade class, you find that the children are having difficulty comparing fractions such as \( \frac{1}{4} \) and \( \frac{1}{3} \). Explain how a quotient interpretation of fractions could help them see that \( \frac{1}{3} \) is larger than \( \frac{1}{4} \).

9. Alvin concluded that \( \frac{1}{4} \), \( \frac{2}{5} \), and \( \frac{3}{9} \) were equivalent and that \( \frac{1}{2} \) was larger than \( \frac{3}{7} \), \( \frac{5}{6} \), \( \frac{7}{9} \), or \( \frac{7}{10} \). What can you conclude about Alvin’s understanding of equivalent fractions? What is Alvin’s fraction-ordering error?

10. Encouraging children to construct models of nonunit fractions using unit fractions (e.g., letting a light-green rod equal the whole and showing one-sixth with a white rod; two-sixths by adding another white rod; three sixths, by adding yet another white rod; and so forth is important for what reasons?

11. (a) Briefly explain why qualitative-reasoning tasks are important both as an instructional tool and an assessment tool. (b) Illustrate your arguments by describing how a teacher could use such a task to teach or assess a particular concept.

12. According to the Student Guide, a fair-sharing analogy is a useful way for helping children make sense of fraction notation. Peck and Connell (1991) reported giving elementary children notation such as: \( \frac{1}{2} \). Though it might seem impossible to make sense of such notation using a fair-sharing analogy, some children found a way to make the analogy work with this type of complex fractions. Explain how the fraction above could be related to a fair-sharing analogy.

PROBLEMS

Fractions in Other Bases (6-8)

How would the darkened part in each of the pictures below be represented as a fraction in (a) base six and (b) base 12?
A. B. C. D. E. F.

\[ \begin{align*} \text{A.} & \quad \text{B.} & \quad \text{C.} \\ \text{D.} & \quad \text{E.} & \quad \text{F.} \end{align*} \]

**Three Shares of Candy** (6-8)

As a snack for their overnight hike into the woods, three Scouts bought a bag of candy. Because each chipped in a third of the cost, they agreed to divide the candy evenly among them. The hike was difficult and the Scouts arrived at their campsite late at night. They were so exhausted, they went right to bed. Early the next morning, the first Scout woke and was famished. Not trusting his firemaking and cooking skills, the Scout found the bag of candy. Honoring his agreement, he counted the candies and took one-third. Still tired, he decided to go back to sleep. Unfortunately, he stepped on the second Scout as he re-entered the tent. The second Scout, now aroused, realized that he was starved. He found the bag of candy. Not realizing the first Scout had already taken his share, the second Scout counted the candies in the bag and took a third. After a healthy belch, he decided to try to get some sleep. The horrendous belch woke up the third Scout who realized that he was terribly hungry. Unaware that his friends had already taken their share, the third Scout counted the candies and took a third. This left eight candies in the bag. How many candies were there to begin with?

**Burned by a Bad Investment** (6-8)

Rogelio got a "hot" stock-market tip from a fraternity brother who had Wall Street connections. A new and extremely promising stock, Burma Amalgamated Mines (BAM), had just come onto the market. Rogelio scraped together every cent he could. Sure enough in the first week BAM doubled in value. Unfortunately, BAM hit water, undrinkable water at that. With their mines flooded, the value of BAM stock dropped like a rock. When Rogelio first heard about the financial straits of BAM, he had already lost \( \frac{2}{3} \) of his original investment. By the time he was able to get to a telephone to call his investor, he had lost another \( \frac{4}{5} \) of what was left of his original investment. Because of a breakdown in the long-distance phone system, Rogelio's stock dropped \( \frac{4}{5} \) of its remaining value just in the time he was on the telephone. By the time he contacted his stock broker and could unload his BAM stock, Rogelio had only $50 left of his original investment. How much had he invested originally?

**Leftover Roast** (6-8)

Mrs. Cook bought a roast. Her family ate \( \frac{3}{4} \) of it for dinner, leaving \( \frac{3}{4} \) of a pound. How much did the roast weigh when Mrs. Cook bought it? (Assume there was no weight loss due to cooking).

(a) Is this problem more like a missing-fraction, missing-part, or a missing-whole (reverse-processing) problem? (b) What is the value of the missing element?

**Mary's Socks**

Consider the following word problem: One-half of the socks in Mary's top drawer are white, and one-third in the bottom drawer are white. What fraction of all Mary's socks are white? A group of students came up with the answer of \( \frac{5}{6} \).

(a) Why is \( \frac{5}{6} \) not a plausible answer? (b) One group of students argued that \( \frac{5}{6} \) would be plausible if the number of socks in each drawer were equal. Evaluate this conclusion. (c) What question does the answer \( \frac{5}{6} \) answer?

\[ \text{This is based on a similar problem that appears in Introduction to Fractions, Book I, A Peatmoss Book by Donald M. Peck and Michael L. Connell (1990) published in Salt Lake City by the Wild Goose Co.} \]
ANSWER KEY for Student Guide

Key for Probe 9.1 (page 9-3)

1. a:C, b:C, c:D, d:C, e:C, f:D, g:D(?), h:C. Note that time, weight, pressure, and elevation are quantities that might involve parts of units and that are measured. A population figure or car inventory involve discrete objects which are countable. The size of the national debt rounded off to dollars is a discrete quantity. If we talk about it in terms of dollars and cents, then we have introduced partial dollars (which are theoretically a continuous quantity). The level of math anxiety is not something that can be counted but something that ranges from none to death-inducing panic.

2. (a) The number of eggs in the carton is a discrete quantity: it is a collection of countable (three) things. (b) The weight of eggs is a continuous quantity. (c) The volume of an egg is a continuous quantity. (d) The depressions are all connected, but the number of depressions can be viewed as a set of (12) countable things, and thus, as a discrete quantity. (e) The amount (volume) of water that can fill the depressions would be a continuous quantity. (f) No, the egg carton can be viewed as either a discrete quantity (How many egg cartons do you see?) or as a continuous quantity (What is the weight, volume, surface area, tensile strength, or density of the carton?).

Key for Investigation 9.1 (pages 9-4 and 9-5)

Part I

Problem 1. By subdividing the acre into equal parcels, it is clear that Mr. Bennett owns 3 of 16 parcels. Using whole numbers, this solution might be recorded as 3 of 16 equal-sized subdivisions of an acre. With fractions, we can record the solution more simply as \(\frac{3}{16}\) acres.

Problem 2. Three of seven equal-size (one-quarter-inch) portions; \(\frac{3}{7} \times 100 = 43.05\) years.

Problem 3. Ten of 30 equal-sized (quarter-inch) portions; \(\frac{1}{3} \times \$3.60 = \$1.20\).

Part II

There various ways for dividing a straw into two equal pieces without folding it. For example, use a ruler. If a ruler is not available, one could estimate the midpoint and mark off the beginning and end of one piece on a piece of paper or string and check this length against the length of the other piece. As necessary adjust the marked off length until each part of the straw is the same. For five equal shares, one could estimate and mark off about one-fifth of the straw length on a piece of paper or a straw. Use this to mark off "fifths" on the straw. If all the pieces are not the same length, adjust your estimate. Note that in all cases, the goal is equal partitioning of the whole.

Sidelight: 1. \(\bigcap\) 2. \(\bigcup\)

Part III

1. (a & b) Unlike other operations, division with whole numbers does not always result in a whole-number answer.

2. For B, the leftover cannot be subdivided, and so the solution would be represented \(10 \div 3 = 3\, r1\). For D, it could be, and so the solution can be represented by \(10 \div 4 = \frac{22}{4} = 2\frac{1}{2}\).

3. (a) Charlie probably viewed the numbers as representing a discrete quantity. (c) Where the numbers represent measurements.

Key for Probe 9.2 (pages 9-8 and 9-9)

Part I

1. Fair-sharing problems imply divvying up (division) and, thus, a quotient subconstruct. For example, four pizzas shared among five people implies four divided among five and can be represented by \(4 \div 5 = \frac{4}{5}\). Each person’s share would be \(\frac{4}{5}\) of a pizza and involves a part-of-the-whole meaning.

2 to 4. These questions involve a part-of-a-whole or fractional-measures subconstruct: shaded

\[\text{\footnote{Some mathematics educators distinguish among continuous, countable-continuous, and discrete quantities. A countable-continuous quantity is a continuous quantity whose subdivisions are treated as a discrete (countable) quantity. For example, the depressions in an egg carton are all connected but they can be viewed as a collection of separate entities. Likewise, the pieces of pizzas are parts of a pizza but can be viewed as a collection of discrete entities and, thus, countable.}}\]
parts relative to total number of parts (whole). Other interpretations are possible. Question 4, for instance, could also be viewed as a rate or a ratio problem (color one of every three).

5. Operator meaning ($\frac{3}{5} \times $2835 = withholding).

6. This question involves the relative magnitude of two quantities (3 Chevies for every 4 Fords) and, thus, illustrates a ratio subconstruct.

Part III

Rational numbers are different from the whole numbers in some important ways.

1. d (an infinite number). A whole number such as 5 may be represented in various ways (e.g., ..., $\sqrt{5}$, 3 + 2, 7 - 2), but most often it is represented by the symbol 5. A particular rational number, though, can be represented by an infinite number of fractions (e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = ... \frac{20}{100} = ... \frac{502}{1004} = ...$). A family of fractions takes its name from the simplest term: the fraction with the smallest numerator and denominator. For example, the following equivalent fractions $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, ...$ are called one-half.

2. Whole numbers represent absolute values: how much there is of something. For example, 12 can represent a dozen pencils, and this is a greater number of things than eleven things. Fractions, though, represent a relationship and—by themselves—may not specify an actual or absolute value. If there are 12 boys in a class of 24 children, then we can say that the fraction of boys in the class is $\frac{12}{24}$. In this case, the fraction does specify the size of the part and the size of the whole as well as the relationship between the part and the whole. However, $\frac{12}{24}$ is another name for $\frac{1}{2}$, and this fraction does not specify the actual size of the part or the whole. Indeed, $\frac{1}{2}$ can also represent any of the following situations: 2 boys in a class of 4, 6 boys in a class of 12, or 15 boys in a class of 30. In brief, a fraction by itself may specify only a relative value. If both a fraction and the size of a whole are known, then an absolute value can be ascribed to a fraction. For example, $\frac{1}{2}$ of the class of 24 has a particular value: 12. The upshot of this is that Question 2 is unanswerable. One-half is greater than one-third if the fractions refer to the same whole. However, there is no guarantee that Jasper and Janine collected the same amount of candy. And if Jasper collected 12 pieces of candy and Janine collected 21, then $\frac{1}{2}$ of 12 is actually less than $\frac{2}{3}$ of 21.

3 & 4. On any given segment of a number line there are a specified number of whole numbers but any number of fractions. Thus, it is more probable that a randomly chosen point will be a fraction of a whole than an integer. Moreover, you can always find another fraction of a whole between any two rational numbers. For example, $\frac{7}{24}$ is between $\frac{1}{4}$ and $\frac{1}{3}$ and $\frac{13}{48}$ is between $\frac{1}{4}$ and $\frac{7}{24}$, and so forth. In brief, there are an infinite number of fractions between $\frac{1}{2}$ and $\frac{1}{4}$. Because any number of rational numbers crowd the space between two points on a number line, such numbers are said to be dense. (They are not called dense because they are difficult to comprehend.)

5. With the whole numbers, the next number just after or just before is easily determined from the counting sequence. Because rational numbers are dense, you cannot determine what comes just after or before it.

Questions for Reflection

1. (a) Step 4. Two parts (one-thirds) are combined (composed) to form the larger subunit two-thirds. (b) Step 2.

2. There are three levels of units: (a) the whole circle (unit), (b) each third (subunit), and (c) two-thirds (larger subunit). Note that each level can be viewed as a unit in its own right.

Key for Probe 9.3 (page 9-10)

2. b. A description of the informal solution processes and their symbolic representations illustrated in Figure 9.3 (on page 9-10 of the Student Guide) are:

- **Solution A.** Each child gets one whole pizza. The three remaining pizzas were then divided in two and each child was allotted a half. The remaining half was then divided into 5 pieces ($\frac{1}{2}$ of $\frac{1}{2}$ or $\frac{1}{10}$).
In effect, \( 8 \div 5 = 1 + \frac{1}{5} + \frac{1}{10} = 1 \frac{3}{10} \). Note that the last step (considering a fraction of a fraction) may be difficult for primary-age children to comprehend and, as a result, they may not successfully complete this step. For such children, it is enough that they recognize that each child would get a little more than one and one half pizzas—one whole pizza, half of another and a little part of another half.

**Solution B.** Another informal strategy is to allot each child one whole pizza and then divide the remaining three into five equal pieces: \( 8 \div 5 = 1 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 1 + \frac{3}{5} \) or \( 1 \frac{3}{5} \).

**Solution C.** A third strategy is to divide each of the 8 pies into fifths, which literally yields eight one-fifths \( 8 \div 5 = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 8 \times \frac{1}{5} = \frac{8}{5} \).

c. By sharing and discussing their informal solutions, children might recognize that \( 1 \frac{1}{10} = 1 \frac{2}{5} = \frac{8}{5} \).

3. The expression \( 14 \div 3 \) can be thought of as, for example, 14 pies shared fairly among three children. Each child would get 4 pies and two would be leftover (4 r2). If the two remaining pies were subdivided into three parts each, and shared, each child would get \( \frac{1}{3} \) of each remaining pie or another \( \frac{2}{3} \) altogether. Thus, each would get \( 4 \frac{2}{3} \) pies.

**Key for Investigation 9.2: Questions for Reflection** (pages 9-16 and 9-17)

1. Using the “whole” instead of “one” avoids using one to refer to a whole (one whole) and a fractional part of a whole (e.g., one-third).

2. The activity uses nonexamples to underscore the point that fractional parts must be equivalent (the same size).

3. (a) If children are asked to state their answers, Activity 2a would be an example of Phase 2A (verbally labeling a concrete model). (b) If children were required to write their answers, it would be Phase 3A (symbolically representing a concrete model) activity.

4. Mathematically, six 1-cm white rods (six-twelfths), three 2-cm red rods (three-sixths), or two 3-cm light-green rods (two-fourths) could also represent one-half.

5. (a) Activity 2 gives the part and the whole and requires finding the missing fraction. Activity 3 gives the whole and the fraction and requires finding the missing part. Activity 4 gives the part and the fraction and requires finding the missing whole. (b) The tasks in Activity 2 are probably the most common. (c) The (reverse processing) tasks in Activity 4 is probably the most challenging.

6. Deductive reasoning.

7. Activity 6 encourages children to think about equivalent fractions and, hence, lays the groundwork for finding a common denominator, which is important for adding unlike fractions. (To show one-half and one-third of the same whole, select the dark-green rod to represent the whole, the red rod to represent \( \frac{1}{3} \), and the light green rod to represent \( \frac{1}{2} \).)

8. (a) Ideally, Activity 7f will help children discover that six-sixths is the same number as (another name for) a whole. Building up fractions from unit fractions also underscores the multiplicative nature of fractions (e.g., \( \frac{5}{6} \) is literally \( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \) or five one-sixths). (b) The purpose of Activities 7g and 7h is to introduce complex fractions: seven-sixths and nine-sixths. Ideally children will recognize that seven-sixths, say, is equivalent to the mixed fraction one whole and a sixth. Activities 7f to 7h together introduce children to fractions equal to one and more than one. This is important because often times children only see examples of fractions less than one and some may conclude (induce) incorrectly that fractions are always smaller than one.

9. A teacher could show students concrete models briefly and ask them to estimate the fraction represented.

10. Equivalent fractions.
Key for Investigation 9.3 (pages 9-20 to 9-23)

Activity I

1. (a) and (c) are greater than 1; (d) might be depending on what w, x, and y represent.

2. (a) \(\frac{3}{y}\), (b) \(\frac{y}{x}\), (c) \(\frac{x}{y}\)

Activity IV

3. This situation again requires students to draw on the conceptional basis of fractions: Divide the rectangles (wholes) up so that both have the same-sized parts.

4. Use three equally spaced horizontal lines to cut the top rectangle into twelfths; use two equally spaced vertical lines to cut the bottom rectangle in twelfths also.

5. \[
\begin{array}{c|c}
\frac{1}{3} & \frac{2}{6} \\
\hline
& \\
\end{array}
\]

6. Step 1:

\[
\begin{array}{c|c}
\frac{2}{3} & \frac{3}{4} \\
\hline
\frac{2}{3} & \frac{3}{4} \\
\end{array}
\]

Step 2:

\[
\begin{array}{c|c}
\frac{2}{3} = \frac{8}{12} & \frac{3}{4} = \frac{9}{12} \\
\end{array}
\]

Because \(\frac{3}{4} = \frac{9}{12}\), \(\frac{2}{3} = \frac{8}{12}\), and \(\frac{9}{12} > \frac{8}{12}\), it logically follows that \(\frac{3}{4} > \frac{2}{3}\).

7. \[
\begin{array}{c|c|c}
\frac{1}{3} & \frac{1}{2} & \frac{2}{6} \\
\hline
\frac{2}{2} & \frac{4}{6} & \frac{5}{6} \\
\end{array}
\]

Note that \(\frac{2}{2}\) (two-thirds of a half) = \(\frac{2}{3} \times \frac{1}{2} = \frac{2}{6}\) (two-sixths of a whole).

KEY FOR A SAMPLE OF CHILDREN'S LITERATURE

The Teacher from the Black Lagoon (page 9-28)

Mrs. Green’s lesson had one redeeming feature: She illustrated the verbal label *one-half* with a concrete model that surely captured the class' attention. The following is a *partial* list of her lesson's weaknesses:

1. She provided no prefraction partitioning experiences.

2. She did not link the concrete model of half and the verbal term "one-half" to the written symbolism \(\frac{1}{2}\) before assigning homework.

3. Her model was not symmetrical; the top and bottom "half" of the boy's body were probably not the same size.

4. She did not use nonexamples of one-half.