TEACHING TIPS

AIMS AND SUGGESTIONS

Because traditional instruction focuses on written arithmetic, many students may not appreciate the importance of mental mathematics in everyday life. Investigation 7.1: Uses of Mental Arithmetic (page 7-2 in the Student Guide) was designed to help them consider the invaluable role of mental math in our electronic age. Actually classifying the students’ uses of mathematics as predominately or typically Written Math, Mental Math, or Calculator Math (as suggested by Part I of the Investigation) and listing them on a chalkboard (as shown in the bulletin board on page 7-1) can help underscore why the NCTM (1989) has recommended putting more emphasis on mental math and less emphasis on memorizing written algorithms. Analyzing the examples of calculator errors (Part II) can help underscore the importance of fostering number sense and estimation ability.

Unit 7•1: Number Sense

Because the traditional skills approach does not focus on fostering an intuitive feel for number, most products of this approach may not be familiar with the concept of number sense or have a well-developed number sense themselves. The primary aim of this unit is to help readers understand what number sense is, why it is important, and how it can best be fostered. (In regard to the last point, instructors may well find it necessary to emphasize the point that number sense cannot be imposed through direct instruction but must be encouraged indirectly through actual experiences with numbers and reflection on these experiences.) Perhaps the best way to achieve these aims—and to foster their own number sense—is to involve adult students in worthwhile tasks that entail using number sense. Investigation 7.2: Thinking About Big Numbers (page 7-8 of the Student Guide), Investigation 7.3: Qualitative Reasoning About Operations on Whole Numbers (page 7-9), and Investigation 7.4: Whole-Number Operation Sense (pages 7-10 and 7-11)—as well as Investigations 7.A (Target-Number Activity), 7.B (Using Benchmarks to Make Estimates), 7.C (Cross-Out), and 7.D (Exploring Estimation Strategies) on pages 177 to 180 of this guide—provide numerous examples of challenging and interesting tasks that can engage children in thinking about numbers and their behaviors. Investigation 7.5: Choosing an Appropriate Calculation Method (page 7-12) can serve to highlight a key aspect of number sense: the need to thoughtfully evaluate a task and flexibly decide whether a paper-pencil algorithm, calculator, mental computation, or estimation is the most useful tool.

Unit 7•2: Estimation

A great many students have had little estimation training and what training they have had focused on rote memorizing and inflexibly applying a set of rules for rounding off numbers. Part I of Probe 7.1: Rounding Instruction (page 7-16 of the Student Guide) can help underscore the point that successful estimation instruction—instruction that promotes adaptive expertise instead of routine expertise—builds on what children already know (e.g., their counting-by-ten knowledge).

Part II of this probe is intended to create cognitive conflict about the prevalent school-taught rule for rounding: with a 5, round up. The aim is that students will recognize that, for example, 3.5 is no closer to 4.0 than it is to 3.0—that there is not a clear justification for rounding up in every situation. This typically is a revelation to most pre- or in-service teachers. Investigation 7.6: Evaluating Estimates (page 7-17 of the Student Guide) can serve to further underscore that what estimation strategy is chosen depends on the goals of a task. More specifically, even though the school-taught rounding rule produces the most accurate estimate, rounding up (overestimating) can be effective in situations where shoppers do not want to overspend the money they have.
Investigation 7.7: Using Everyday Knowledge to Make Estimates can serve to highlight the importance of thoughtfully and flexibly using existing knowledge to make estimates. This activity illustrates how scientists and others make real estimates.

Unit 7•3: Mental Calculation

Typically, products of the traditional skills approach inflexibly use the standard right-to-left written algorithm for doing multidigit mental computations. Probe 7.2: Strategies for Calculating the Sums and Differences of Multidigit Expressions (page 7-25 of the Student Guide) was designed to help such students recognize that a left-to-right strategy—a strategy many children informally invent—is more efficient in many situations. Investigation 7.E: Exploring Strategies of Mental Computation (page 181 of this guide) can further extend their repertoire of mental-computation strategies and their appreciation for flexibly choosing among a variety of strategies.

Probe 7.3: Analyzing a Game Involving Mental Computation (page 7-28 of the Student Guide) and Investigation 7.F: Card Games 99 and 999 (page 182 of this guide) can help students see that math games can be useful in promoting the development of mental-computation strategies or in practicing mental-computation skills. Consider Grid Race (Activity File 7.4 on page 7-28). To add 10, novices typically count on 10 individual spaces (e.g., from space 14 on the 99 grid, count off 10 spaces beginning with space 15 and ending with space 24). In time, they recognize a shortcut: To add 10, just move up one row (e.g., from space 14, move directly to space 24). This can form the basis of a plus-ten rule: When 10 is added to a two-digit number, the sum is the next decade plus the original ones-place digit. The Card Game 99, likewise, can provide an entertaining and purposeful context for inventing and practicing mental-arithmetic shortcuts. Perhaps the best way to illustrate the value of this game is to actually have students play it and then discuss their experience.

SAMPLE LESSON PLANS

Project-Based Approach

Using SUGGESTED ACTIVITIES 1, 6, 7, 9, 10, 11, 12, and 13 (on pages 183 to 185 of this guide) as a menu, have small groups of students choose a project. Note that most of these activities involve actually working with children. After completing the project, have the groups share their results with the whole class.

Single-Activity Approach

One whole class period, if not more, could be spent on Investigation 7.4: Whole-Number Operation Sense (pages 7-10 and 7-11 in the Student Guide). This reader inquiry could serve as the basis for defining number and operation sense, discussing why they are important, illustrating worthwhile tasks for fostering number/operation sense, and discussing how this ability is critical for estimation and multidigit computation. Note that some of the tasks involve making estimates (Task I in particular) and making mental calculations (e.g., Tasks II to VI). Task I can serve to illustrate, for example, how rounding and calculational shortcuts can provide an efficient basis for estimation (e.g., 135 x 1,986 can be rounded off to 100 x 2000, and the "add two zeros when multiplying by 100" rule can then be applied to arrive at an estimate of 200,000).

Task II can involve either estimation or mental calculation in evaluating the validity of each conclusion. Note that some students may be confused by this task because they assume that the conclusions are premises. For Example 1, the premise is The sum of three two-digit numbers is less than 90. The first conclusion (a) is: Each addend is less than 30. This conclusion may be true: The addends could be 29, 29, and 29, which sum to 87. However, the counterexample 27 + 29 + 31—which is also equals 87 and, thus, consistent with the premise—illustrates that the conclusion that each addend is necessarily less than 30 does not follow. In brief, the conclusion is possibly true. Some students, however, turn the reasoning task around and use the conclusion as the premise: If no addend is greater than 30 (e.g., 29 + 29 + 29), then the sum is necessarily (always) less than 90. This incorrect conversion leads students to conclude that the “conclusion” is necessarily (always) false.

Multiple-Activities Approach

To provide a relatively broad overview of the topic, an instructor might elect to do the following sequence of activities:

1. Begin class with Part I of Investigation 7.1: Uses of Mental Arithmetic (page 7-2 of the Student Guide). While the students are listing their everyday uses of mathematics, the instructor can
put the following headings on a chalkboard: **Written Math**, **Mental Math**, and **Calculator Math**. Have the students share their ideas; have the class vote on whether a use is predominantly written, mental, or calculator math; and write the use under the appropriate heading. Typically, mental math has far more entries than written math. This can help underscore why the NCTM emphasizes mental mathematics.

2. To provide a basis for discussing number sense and operation sense, have the class complete, for instance, **Task III** (A Million vs. a Billion) of **Investigation 7.2: Thinking About Big Numbers** (page 7-8 of the **Student Guide**) and **Task II** (Alphabet Arithmetic) of **Investigation 7.3: Qualitative Reasoning About Operations on Whole Numbers** (page 7-9 of the **Student Guide**). Because students will probably be unfamiliar with the term qualitative reasoning, ensure that they read the introduction to Investigation 7.3 carefully before beginning Task II of this reader inquiry. An instructor may wish to note that, because qualitative-reasoning tasks require conceptual understanding, they are an ideal task for beginning instruction on a topic and for assessing understanding afterward.

3. To underscore the importance of instruction that builds on what children know and that fosters thoughtful and flexible use of estimation strategies, a class can complete **Probe 7.1: Rounding Instruction, Investigation 7.6: Evaluating Estimates**, and **Task 2** (A Big Distance) of **Investigation 7.7: Using Everyday Knowledge to Make Estimates** (pages 7-16, 7-17, and 7-19, respectively, in the **Student Guide**). It may be helpful to do Task 2 as a class, as students typically need considerable guidance. If they do not spontaneously offer a strategy for proceeding, prompt them by asking them what facts about geography they know. Often, someone will estimate that the distance across the United States is about 3,000 miles. If need be, then ask the class to estimate the distance across the Atlantic Ocean, Europe, Asia, and the Pacific Ocean. (See page 190 of this manual for another solution strategy.)

4. To prompt a discussion of mental-addition strategies, particularly left-right strategies, have students complete **Probe 7.2: Strategies for Calculating the Sums and Differences of Multidigit Expressions** (page 7-25 of the **Student Guide**) and **Probe 7.3: Analyzing a Game Involving Mental Computation** (page 7-28 of the **Student Guide**) or **Investigation 7.F: Card Games "99" and "999"** (page 182 of this manual). The latter can be most entertaining as well as educational.

**SAMPLE HOMEWORK ASSIGNMENTS**

Read: Chapter 7 in the **Student Guide**.

**Study Group:**

- **Questions to Check Understanding**: 1 and 2 (page 185).
- **Writing or Journal Assignments**: 2, 3, and 6 (pages 186 to 188).
- **Problem**: Alphabet Addition (page 188).
- **Bonus Problem**: A Hefty Hike or Word Limit (page 188).

**FOR FURTHER EXPLORATION**

**ADDITIONAL READER INQUIRIES**

**Investigation 7.A** (page 177)

**Target-Number Activity** illustrates how an entertaining game can serve as a basis for purposeful computational practice and fostering operational sense. It could serve as an extension for **Investigation 7.2, 7.3, or 7.4** in Unit 7•1 of the **Student Guide**.

**Investigation 7.B** (page 178)

**Using Benchmarks to Make Estimates** could prompt students to discover the estimation strategy discussed in **Box 7.2: Numerical Benchmark Problems** on page 7-23 of the **Student Guide** (see page 191 of this guide for details).

**Investigation 7.C** (page 179)

**Cross-Out** can serve to prompt the discovery and discussion of various computational estimation strategies, such as those discussed on page 7-23 of the **Student Guide** (see page 191 of this guide).

**Investigation 7.D** (page 180)

**Exploring Estimation Strategies** was designed to encourage students to devise an effective estimation strategy for different situations. (See page 191 of this guide for a more complete discussion.)
Investigation 7.E (page 181)

Exploring Strategies for Mental Computation was designed to help students recognize that there are a variety of mental-computation strategies and that an expert calculator’s choice of strategy depends on what would be the most efficient procedure for the numbers involved.

Investigation 7.F (page 182)

Card Games "99" and "999" describes two entertaining games for practicing mental computation with two-digit and three-digit addends, respectively.

QUESTIONS TO CONSIDER

1. You point out to your student teacher that there is no one correct way of rounding with 5—that how you estimate depends on the requirements of the tasks. Your student teacher seems quite worried about this advice. "What do I do if I have to give a standardized test that requires kids to round 5 up?" she asks. How would you respond?

2. A key implication of Piaget’s theory is that the pace of cognitive development limits what mathematical concepts can be learned by children (e.g., Kamii, 1985). For example, young children typically can’t solve missing-addend problems (e.g., change-add-to unknown-change problems) or missing-addend expressions such as $5 + \square = 9$ or $\square - 3 = 7$. This has been attributed to their lack of part-part-whole knowledge—the conceptual basis for understanding and solving such tasks (e.g., Riley & Greeno, 1988; Riley, Greeno, & Heller, 1983).

A recent study (Sophian & McCorgray, 1994) gave 4-, 5-, and 6-year-olds change problems with an unknown start like those below.

- **Problem 7.A: Change Add To, Unknown Start (K-2)**. Blanca bought some candies. Her mother bought her three more candies. Now Blanca has five candies. How many candies did Blanca buy?

- **Problem 7.B: Change Take Away, Unknown Start (K-2)**. Angie had some pennies. She lost two pennies while playing. Now she has seven pennies. How many pennies did Angie have before she started to play?

Although 5- and 6-year-olds typically had great difficulty determining the exact answers of such problems, they at least gave answers that were in the right direction. For Problem A, for instance, children knew that the answer (a part) had to be less than five (the whole). For Problem B, for example, they understand that the answer (the whole) had to be larger than seven (the larger of the two parts).

(a) What do the results of this study suggest about 5- and 6-year-olds operation sense about missing-addend word problems? (b) Are these results consistent with the hypothesis that an inability to solve missing-addend word problems is due to a conceptual deficiency—the lack of part-whole knowledge? (c) Why might 5- and 6-year-olds be unable to determine the exact solutions to missing-addend word problems?

3. (a) Using $34 + 25$ as an example, model a partial decomposition strategy with base-ten blocks. (b) Compare demonstrating a partial-decomposition strategy with base-ten blocks versus demonstrating the strategy with a 0 to 99 grid like that on page 7-28 of the Student Guide. Which demonstration would more elegantly illustrate the power of the partial decomposition strategy? Why?

4. The IRS once instructed taxpayers to round down (e.g., $93.50$ could be recorded as $93$). Now they instruct taxpayers to round up (e.g., $93.50$ must now be recorded as $94$). If this rounding procedure is used for listing taxable income, deductions, and amounts withheld (amount already paid in), what are the implication of this change in policy? For most taxpayers, would this significantly increase, decrease, or have a negligible effect on the taxes owed?

5. (a) Which of the following rounding strategies (described in Question 3 of Investigation 7.D on page 180 of this guide) would give the most accurate estimate of $23 \times 104$: rounding both factors down, rounding just the smaller factor down, or rounding just the larger factor down? Why? (b) Are any of these strategies more accurate than rounding one factor up and one factor down? Why or why not? Use $28 \times 47$ as an example to explain. (Text continued on page 183.)
Investigation 7.A: Target-Number Activity

- Computational practice + number sense
- 1-8
- Any number

In this activity, students start with a given number (e.g., 22), and repeatedly choose between two arithmetic functions (e.g., +2 and -4) in an effort to change the starting number into a target number (e.g., 10). For each step, a child may choose only one of the functions. The object of the game is to achieve the target number in the fewest steps possible. Students should record each step in permanent ink so that they cannot change their solution after discovering a shortcut.

<table>
<thead>
<tr>
<th>START: 48</th>
<th>START: 28</th>
<th>START: 27</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHOICES: +2 or -10</td>
<td>CHOICES: -2 or -10</td>
<td>CHOICES: x2 or -10</td>
</tr>
<tr>
<td>TARGET NUMBER: 10</td>
<td>TARGET NUMBER: 4</td>
<td>TARGET NUMBER: 13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step</th>
<th>Step</th>
<th>Step</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>19</td>
<td>19</td>
</tr>
</tbody>
</table>

†Source: Creating problem-solving experiences with ordinary arithmetic processes by J. Bernard and published in *Arithmetic Teacher, Vol. 30* (No. 1) [September 1982], 52-55.
Investigation 7.B: Using Benchmarks to Make Estimates

◆ Set-size estimation ◆ 1-8 ◆ Any number

This investigation involves presenting a class with, for example, an 8-inch x 11-inch page covered with a haphazard array of dots. Challenge them to gauge the number of dots. (The page should contain enough dots to make counting impractical.) Try this task yourself with the haphazard array below. What strategy would be helpful in determining the total number of dots? Share your strategy and estimate with your group or class.

Questions for Reflection

1. How could the task above be adapted for first graders?

2. Consider how a class could estimate (a) the number of dandelions in a lawn, (b) the number of stars in the milky way galaxy, and (c) the number of bricks in a brick facade of a building.
Cross-out is an entertaining way to introduce computational estimation and to foster discussion about a variety of estimation strategies, including rounding. The game is recommended for second grade and up. The game begins with a teacher writing a number sentence like those below on the chalkboard. Teams of four children decide which term should be crossed out to make the number sentence true. After reaching a consensus, a group can record its answer on a miniature chalkboard or a piece of paper. After a time limit (e.g., 20 seconds), each group holds up its answer, which is then recorded on a chalkboard. The teacher has the class discuss the answers and the strategies used. After the class agrees on the correct answer, the results of the round are then scored. The scoring can be done in various ways. One method is to award 2 points to the first group(s) that raised a hand and was correct. One point can be awarded any group that was correct within the time limit: (A time limit underscores the need to answer quickly and use estimation rather than computational strategies.) A commensurate number of negative points could be scored if a group responded incorrectly. This would serve to introduce and practice negative numbers and to discourage wild guessing. (A team that raised its hand first but was incorrect would get -2 points.)

Either playing against other individuals in your group or other groups in your class, try the game using the number sentences below. Note the various strategies used for each item and which is most efficient.

a. $7 + 7 + 15 + 35 = 29$
b. $9 + 8 + 46 + 39 = 56$
c. $2 + 1 + 33 + 48 = 51$
d. $15 + 35 + 42 + 95 = 92$
e. $18 + 37 + 59 + 38 = 134$
f. $25 + 37 + 21 + 45 = 91$
g. $23 + 34 + 35 + 40 = 97$
h. $33 + 27 + 36 + 21 = 84$
i. $45 + 34 + 27 + 26 = 106$
j. $47 + 56 + 75 + 87 = 209$
k. $103 + 46 + 109 + 85 = 258$
l. $125 + 57 + 76 + 110 = 311$
m. $150 + 103 + 147 + 175 = 400$
n. $149 + 151 + 100 + 125 = 425$
o. $199 + 99 + 299 + 399 = 697$
Investigation 7.D: Exploring Estimation Strategies

Discovering estimation strategies 1-8 Any number

The aim of this investigation is to prompt students to invent effective strategies for various estimation tasks. To understand what is involved and to perhaps expand your own repertoire of estimation strategies, answer the questions below. Share your answers and strategies with your group or class.

1. Mr. Poke's son Shad was a menace. The boy had stolen a chemical from the middle-school science laboratory and had thrown it into a toilet in the boys' room. The resulting explosion caused massive damage. Mr. Poke was presented with the following itemized bill for repairs and materials:

   wall repairs ......................$432
   window repairs ...................$87
   electrical repairs ................$296
   light bulbs ................................$8
   new stall ..............................$225
   new toilet ..........................$156
   new mirror .......................... $32

   Mr. Poke's head reeled. "About how much is that?" he asked feebly.

How could the principal, who was being beckoned by a teacher to attend to another emergency and who did not have time to round off, give Mr. Poke a rough but quick estimate? Hint: Consider what digits have the most impact on the total.

2. Mr. Poke decided to send Shad to a military academy for some discipline. If Mr. Poke had $7,685.76 in his checking account and he wrote a check for $2,875 for tuition, about how much would he have left in his account? Note that in a skills approach, students would be encouraged to round both numbers. What would be a more accurate and easier way of estimating Mr. Poke's balance?

3. Soon after Shad arrived at the military academy, Mr. Poke read in the newspaper that 27 antique cannons worth $4,895 each had somehow been freed of their moorings and had rolled down an embankment into a nearby river. Expecting the worse, Mr. Poke quickly estimated the total cost of the ruined cannons. Note that in a traditional skills approach, students might be expected to round both factors up. Compare this strategy with rounding up just the 27 and with rounding up just the 4,895. (a) Which of the three strategies just mentioned is the most accurate and why? (Hint: In each case, consider how much a factor gains and how many times this rounding error is multiplied.) (b) Is the most accurate strategy in this case always the most accurate of the three strategies? Why or why not? Which strategy is the most accurate?

4. Sure enough, Shad had been behind the cannon escapade. (School officials had become suspicious when Shad had suddenly taken on a sunny disposition and began singing continually, "And the caissons go rolling along.") However, the damage was not as great as Mr. Poke feared. The school commandant read off the charges, $67.50, $32, $81.77, $23.05, $10.99, and $92.35. What would be a relatively easy way to estimate the total damages?

5. After Shad was dismissed from the military academy, Mr. Poke got a call from The Pit, the local teen hangout. Apparently his son Shad had run up the following tabs: $5.42, $4.81, $5.95, $4.02, $5.10, $5.25, $4.86, $4.99, $5.03. About how much did Shad (Mr. Poke really) owe? Other than rounding, what estimation strategy would be particularly useful for making a quick educated guess as to the outstanding tab?
Investigation 7.E: Exploring Strategies for Mental Computation

- Mental multidigit arithmetic - Any number

The aim of this activity is to help students construct efficient mental-calculation strategies. After reading about each of the following classroom activities, try it yourself. Share and discuss your ideas with your group or class.

Part I: Brainstorming About Strategies (↔ 1-8).

Present students with an arithmetic expression, such as 86 - 57 and challenge them to come up with as many different ways of mentally calculating the answer as possible. Encourage students to share their methods and to discuss the advantages and disadvantages of each method.

Part II: Using an Area Model to Do Mental Multiplication (↔ 3-8)

Concrete-Model Phase. Many people believe that mental multiplication beyond 10 x 10 is too hard to do. An area model makes two-digit mental multiplication practical. Consider, for example, 13 x 17. (a) Use base-ten blocks to construct an area model of this expression. (b) Your model should reflect that 13 x 17 = (10 + 3) x (10 + 7). What is the area (product) of the 10-by-10 section (10 x 10), the 10-by-7 section (10 x 7) the 3-by-10 section (3 x 10), and the 3-by-7 section (3 x 7)? (c) Determine the total area (the product) by summing the area of the four sections (the partial products).

Mental-Image Phase. Now imagine an area model of 12 x 15. Using this mental picture, determine the area of each section (each partial product) and then sum them. (Initially, you may wish to record the partial products.) Do the same for 24 x 56. If you want a real challenge, try 16 x 128.

Part III: Discovering Compensation Strategies for Multiplication (↔ 3-8)

Compensation with Eights and Nines. Consider the following expressions in light of a groups-of meaning: (a) 99 x 17, (b) 329 x 6, (c) 98 x 23, (d) 498 x 7, (e) 34 x 199, (f) 42 x 999, (g) 76 x 198, and (h) 62 x 448. What thinking strategy would make the mental multiplication of such expressions much easier?

Compensation for "Nice" Multiples. Maryann explained that she solved 5 x 128 by doubling 5 to make the "nice number" 10 and then to compensate for this adjustment, divided 128 in half to get 64. Next she multiplied 10 and 64 to get the product—640. (a) How would this strategy work for 50 x 72? (b) Would this mental-multiplication strategy work efficiently with expressions such as 5 x 127 or 50 x 73? Why or why not? What does this suggest about the application of this strategy? (c) Consider how Maryann's strategy could be applied to 80 x 25 and 25 x 240.

Part IV: Considering Shortcuts for Multiplication or Division Involving Zeros (↔ 4-8).

Consider how to short-cut mental multiplication by 10, 100, 1000, and so forth. Now consider how 3200 ÷ 80 and 24,000 ÷ 600 could be short-cut in a similar manner.

Part V: Devising Mental-Computation Shortcuts (↔ 3-8).

A key aspect of everyday mental computation is adjusting one's strategies to make computation as easy as possible. Consider the following situations. In each case, try to devise a shortcut to reduce the calculational effort required.

Situation 1: Bad Tax News. To her chagrin, Zoe's accountant noted that she owed $2,998 in Federal taxes, another $1,386 in state taxes. She had just started to make out the check when her accountant had to answer the phone. Not wanting to interrupt an obviously important call and in a hurry to get back to her office before the end of her lunch break, Zoe calculated the sum of 2,998 and 1,386 mentally.

Situation 2: A Bill Adjustment. Ryan checked Mr. Sevier's bill to make sure he wasn't charged for a part that was still under warranty. The bill of $876 did include an undue charge of $98 for the part. She had just started to make out the check when her accountant had to answer the phone. Not wanting to interrupt an obviously important call and in a hurry to get back to her office before the end of her lunch break, Zoe calculated the sum of 2,998 and 1,386 mentally.

Situation 3: Too Many Grades. Miss Brill was having a bad day. She had to add up the math quiz grades for the semester—10 for each child in her class, and she had just stepped on and broken her solar-powered calculator. Consider the sample of scores listed below. What are the sums in each case?

<table>
<thead>
<tr>
<th>Abigail</th>
<th>6</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brynnan</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>Dale</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>
Investigation 7.F: Card Games "99" and "999"

- Multidigit mental addition and subtraction
- 1-8
- Two to six players

After making up a deck of cards as described below and reading the instructions, try playing the following games with your group or some friends. As you play, consider the game’s pedagogical value.

Card Game 99

This enjoyable card game is an excellent way to prompt the invention and discussion of multidigit mental addition and subtraction to 100, particularly that involving +10. It is also useful practicing such mental arithmetic. Briefly, the object of play is to avoid discarding a card that puts the discard total over 99. Three cards are dealt to each player.

- +2 to +9: These cards add their face value to the discard total.
- +1 or +11: Adds 1 or 11 to the discard total.
- Reverse: Adds nothing to the discard total but reverses the direction of play.
- 99: Automatically makes the discard total 99. (It does not add 99 to the discard total or put the player out.)
- -10: Deducts 10 from the discard total.
- +0: Adds nothing to the discard total.
- +10: Adds 10 to the discard total.

The player to the left of the dealer starts by discarding, announcing the new discard total, and drawing a replacement from the deck. (A replacement card must be drawn before the next player draws a replacement card. If a player fails to do this, then he or she must play with fewer cards—a distinct disadvantage in this game.) Play continues until someone loses (puts the discard total over 99). The loser gets one strike. Three strikes puts a player out of the game. With two players, a person wins when the other player goes out of the game. With more than two players, a person wins when all other players are eliminated.

The game is commercially available. However, teachers may wish to use 3 x 5-inch cards to make up their own decks to better suit the developmental level of their students. The cards for a basic deck suitable for first- and second-graders, an intermediate deck suitable for first to third-graders, and an advanced deck suitable for second to third-graders are delineated below:

![Card Game 999 deck diagram](image)

Card Game 999

To practice three-digit mental addition, third-graders can play the Card Game 999. To construct a deck for Basic 999, include about four of each of the following: +0, +10, +20, +30, +40, +50, +100, -10, -100, 999, and Reverse. To construct a deck for Intermediate 999, add to the Basic 999 deck about four of each of the following: +60, +70, +80, +90, +15, +25, +35, +45, and +55. To construct a deck for Advanced 999, add to the Intermediate 999 deck, about 20 cards with two-digit numbers such as, +18, +27, +36, +44, and +53.

Teaching Tips

- When players discard, have them announce the prior total of the discard pile, what they are adding, and the new total. This slows play and gives all players the opportunity to reflect on the reasonableness of the new total.
- Laminating the cards will help them last longer.
SUGGESTED ACTIVITIES

1. Examine newspapers, magazines, or the Internet to find both reasonable and suspicious "fact-based" claims. Evaluate whether the numbers are believable or not. Create a file of such claims that would be appropriate for children at different grade levels to evaluate using a familiar benchmark.

2. (a) Design several What Can You Tell From? items (see Task II of Investigation 7.4 on page 7-10 of the Student Guide). For example, create a true, a false, and a possibly true statement for the following conditions: The sum of three two-digit numbers is more than 100; The difference of two three-digit numbers is less than 100; The product of two two-digit numbers is more than 200; and The product of two two-digit numbers is less than 200. (b) Try them out with children.

3. (a) Choose a grade level and content area such as science, social studies, literature, or music. Design a Fit-the-Facts task (see Activity File 7.6 on page 7-30 of the Student Guide) that would be developmentally appropriate both in terms of the numbers used and the subject content. Check your tasks to ensure that it is not ambiguous—that each blank could have only one possible answer. (b) Try it out with children and revise as need be.

4. Reardon must count to determine the sum of the following multidigit expressions:

\[
\begin{align*}
28 + 2 & \quad 68 + 2 & \quad 88 + 3 & \quad 17 + 3 & \quad 37 + 3 & \quad 57 + 3 \\
16 + 4 & \quad 76 + 4 & \quad 96 + 6 & \quad 24 + 6 & \quad 44 + 6 & \quad 74 + 6
\end{align*}
\]

Design a discovery-learning ("Math-Detective") exercise that could help Reardon exploit his existing knowledge of arithmetic so that he could efficiently determine the sums of the multidigit combinations above.

5. Make up an alphabet arithmetic task (see, e.g., Task II on page 7-9 of the Student Guide) for (a) four-digit addition with renaming, (b) two-digit subtraction with renaming, (c) three-digit subtraction with renaming, and (d) four-digit subtraction. Be sure to check them to make sure they have at least one possible solution.

6. (a) Interview three kindergartners and three first graders. Read them Problems 1, 2, and 6 on page 5-6 of Probe 5.1 in the Student Guide and Problems 7.A and 7.B on page 176 of this guide. If they do not calculate the correct answer in each case, evaluate whether or not it is at least in the right direction. Be sure to ask each child to explain his or her answer. (b) Present the same children with symbolic expressions such as 7 + 1, 8 + 3, 4 + 6, and 2 + 9 and then ask them to estimate the answers. Did the children use systematic estimation strategies? If so, briefly describe the strategy. (c) Make up some word problems involving repeated addition or divvying up. Present them to a number of second and third graders. Ask children to explain their answers. Evaluate the children's operation sense. (d) Select a set of symbolic multiplication combinations and present them to second or third graders. What estimation strategies do they use? Evaluate the children's operation sense. (e) Summarize your data and report it to your class.

7. (a) Devise a lesson that uses the investigative approach to introduce content related to this chapter. The lesson should begin with a problem or some form of inquiry that raises questions or issues leading to the exploration of content covered in this and possibly other chapters. Specify what mathematical content children would have to explore and learn about to solve the problem. Indicate how you would guide children to explore and learn the content in an intuitive and relatively autonomous manner. Indicate also what types of preparatory experiences children should have to solve the problem and explore the content. (b) Implement your lesson with a small group or class of elementary-level children. Assess their response and learning. Evaluate the lesson and indicate how it could be improved.

8. Make up three Target Number activities that would be appropriate for the grade level you plan to teach. For examples, see Investigation 7.A on page 177 of this guide.

9. Choose one or more of the reader inquiries from chapter 7 of the Student Guide or this guide to teach an elementary-level class. Evaluate how the children responded. Indicate what you learned about their number sense. Evaluate the activities. How might you modify them for future use?
10. Administer a diagnostic test such as the TEMA-2 (Ginsburg & Baroody, 1990) to several primary-level students and evaluate their number sense. Applicable items on the TEMA-2 include Items 1 (Intuitive Numbering), 4 (Perception of More), 16 (Conception of More), 23 (Mental Number Line—One-Digit Numbers), 31 (Mental Number Line—Two-Digit Numbers), and 51 (Mental Number Line—Three-Digit Numbers). For each child, summarize what you learned about his or her number sense, make suggestions for further assessment, and recommend needed instruction, including appropriate activities from the Student Guide. Share your results and conclusions with your class.

11. Familiarize yourself with the different types of estimation tasks illustrated in Figure 7.1 below. Using collections of 7, 8, 12, 13, 17, 18, 23, and 24 and benchmarks of 10 and 20 (if applicable), administer Tasks A, B, C, or D to a half dozen children in each of the following grade levels: K, 1, and 2. (a) Evaluate the success of each child’s estimation efforts and summarize your data in a report. Include in the report your conclusions about the effects of set size on children’s estimation ability, any significant changes with age, whether or not children seem to have constructed a mental benchmark of 10, 20, or both. (b) Compare your data to that of others who used the same tasks. Indicate in your report which task was easiest for children, which was next easiest, and so forth. (c) Describe the instructional implications of your results. (d) Share your results and conclusions with your class.

12. Watch one of the following videotapes by Constance Kamii and published by Teacher College Press: (a) Double-Column Addition: A Teacher Uses Piaget’s Theory; (b) Multiplication of 2-Digit Numbers: Two Teachers Using Piaget’s Theory, and (c) Multidigit Division: Two Teachers Using Piaget’s Theory (see page 6-30 of the Student Guide for more information). Analyze the self-invented informal strategies that the children used. Write a report that describes the strategies and whether each was effective or not. Did any children use the standard

---

**Figure 7.1: Number-Estimation Tasks**

In each of the four tasks below, a collection of unknown size is briefly shown to children and then covered to prevent counting. Note that these tasks involve estimating the size of a collection. A student could also be asked to find a collection of a given number. For example, this variation of Task C would entail finding a collection of more than five things.

<table>
<thead>
<tr>
<th>Concrete benchmark present</th>
<th>Concrete benchmark not present</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Benchmark-comparison tasks</strong> (Gauging whether a set is more or less than a benchmark)*</td>
<td></td>
</tr>
<tr>
<td><strong>A. Comparison to a concrete benchmark</strong></td>
<td><strong>C. Comparison with a mental benchmark</strong></td>
</tr>
<tr>
<td><img src="image1" alt="Concrete benchmark present" /></td>
<td><img src="image2" alt="Concrete benchmark not present" /></td>
</tr>
<tr>
<td>This is five. Is this more (less) than five?</td>
<td>Is this more (less) than five?</td>
</tr>
</tbody>
</table>

*Note that an unknown set can be compared to two benchmarks as well as one.

<table>
<thead>
<tr>
<th><strong>Approximation tasks</strong> (Gauging the size of a collection)**</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>B. Estimation using a concrete benchmark</strong></td>
</tr>
<tr>
<td><img src="image3" alt="Approximation tasks" /></td>
</tr>
<tr>
<td>This is five. About how many is this?</td>
</tr>
</tbody>
</table>

**Task D is called open-ended because no restrictions are placed on the child’s answers.**
(right-to-left) renaming algorithm illustrated on page 7-25 of the Student Guide? Did any use the left-right decomposition or partial-decomposition strategy also described on this page? Did any of the children use any of the strategies described in Investigation 7.E (on pages 181 of this guide)?

13. (a) Interview at least four children in each of the following grade levels: second, fourth, sixth, and eighth. Have them read the Hardware Cost problem (on page 7-25 of the Student Guide), solve it, and describe their strategy for solving the problem. (b) Perform an analysis, answer the questions, and write a report as detailed in Activity 12 above. (c) Present your results to your class.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. Circle the letter of any of the following statements that, according to the Student Guide, is true. Change the underlined portion of any incorrect statement to make it true.

   a. Mental arithmetic and estimation, in particular, is less important today because of the ready availability of computers.

   b. Mental arithmetic has traditionally been overemphasized by textbooks and standard tests.

   c. Good estimation skills are necessary for good mental-computation skill but not vice versa.

   d. Fostering number sense depends on promoting routine expertise.

   e. Qualitative reasoning is a basis for understanding arithmetic procedures but not useful in checking computations.

   f. How rounding is implemented depends upon the estimator’s aim or objective.

   g. Number (set-size) estimation should begin with collections of about 30 to 50.

   h. Estimation instruction should begin with estimating the size of collections and measures, not multidigit computations.

   i. Knowledge of the basic number combinations is not central to estimating the sums of multidigit terms.

   j. Estimation instruction should begin with a guess-and-check procedure.

   k. Feedback on estimates should focus on which estimate is closest to the correct answer.

   l. The correct rounded estimate of 384 + 296 + 151 is 900.

   m. A front-end estimate of 7,989 + 976 + 1,021 + 935 + 4,042 would be 13,000.

   n. There are few recurring patterns that children can exploit to learn multidigit number combinations.

   o. Grid Race (see Activity File 7.4 on page 7-28 of the Student Guide) should be played after children have stopped using counting to add 10.

2. The digits in the division expression to the right have been covered up with letters. (a) If the value of the digits under both A and D decreased, what would happen to the size of the quotient? Explain why. (b) If the value of both C and D increased, what would happen to the size of the quotient. Explain why.

3. (a) Why would the qualitative-reasoning tasks illustrated in Investigation 7.3 (on page 7-9 of the Student Guide) be useful as evaluation items as well as instructional tools? (b) The qualitative-reasoning tasks illustrated in Investigation 7.3 involve which type or types of reasoning—intuitive, inductive, or deductive reasoning? Why?

4. For the expression to the right, what estimates would result from (a) rounding to the one-thousands place and (b) using a front-end strategy?

5. A front-end approach would give what es-
(timates for the following two expressions: a) 3,476 x 385 and (b) 6,899 ÷ 124?

6. Preservice teachers were asked to devise a realistic question involving a front-end estimation strategy. Evaluate whether or not each of the following examples successfully addressed the assignment. Briefly explain why or why not. Consider how front-end estimation is done, the relative accuracy of such estimates, whether such estimates result in conservative or liberal estimates, and what everyday situations this strategy would be most appropriate (more useful than other estimation strategies).

a. Argo’s example: Becker has only $10 to spend at the grocery store. To make sure, he does not spend too much, he uses a front-end estimation strategy. If Becker picked up items costing $3.25, $1.99, $2.15, and $2.76, what would be his estimate?

b. Arlette’s example: Zep needed a fairly accurate estimate of how much he had charged on his credit card. If he charged $185, $110, $65, $852, and $75, about how much had he charged altogether?

c. Nyambura example: A mail-order company noted that with a minimum purchase of $50, it would pay the postage for the order. Leanne wanted to order items costing $12, $21, $8, and $24. Would her order be enough to get free postage?

d. Fernando’s example: Rita’s boss needed a really quick estimate of how many thousands of dollars worth of business their company had done that week. Looking at the sales summary for the week, Rita saw that there were $1,236 worth of sales on Monday, $2,342 on Tuesday, $876 on Wednesday, $3,126 on Thursday, and $3,568 on Friday.

7. Illustrate how using compatible numbers can be used to simplify the mental calculation of (a) 4,885 + 2,047 and (b) 3,803 - 1776.

8. Mrs. Fields was astounded that her fourth graders would choose a paper-and-pencil method to determine the difference of such numbers as 350 and 1 (350 - 1), 350 and 10 (350 - 10), 500 and 1 (500 - 1), and 500 and 10 (500 - 10). Even when she asked them to determine such differences mentally, they still relied on using the standard right-to-left (renaming) algorithm. (a) What could Mrs. Fields do to help her students see that other mental-arithmetic procedures were more efficient and more appropriate? (b) What might account for her fourth-graders’ reliance on the relatively inefficient right-to-left standard algorithm in these cases? (c) Specify what counting skills Mrs. Fields should check.

9. One way to establish a range of reasonable answers for a computation is to first round up to determine the upper limit and then round down to determine the lower limit. What other estimation strategy described in the Student Guide could be used to determine the lower limits of a reasonable answer to a computational task?

10. While keeping score for a game (PacMan), Alexi, who already had 56 points, scored another 6. The 9-year-old estimated, “It’s going to be 61 or 62.” How could a teacher relate this task to what Alexi should already know in order to help him decide between 61 and 62?

11. Gina had great difficulty mentally adding two-digit numbers such 40 and 30, or even decades such as 40 and 10. The girl relied on counting on and often miscounted. (a) How could a teacher build on existing counting and arithmetic knowledge to help a child such as Gina to become proficient in adding combinations such as 40 + 30, 70 + 20, 40 +10 and 70 + 10? (b) What counting and arithmetic skills provide the foundation for such a remedial approach and should be checked before remedial instruction begins?

WRITING OR JOURNAL ASSIGNMENTS

1. For one day, keep a diary of how you use mathematics. Classify each entry as (a) written math, (b) computer or calculator math, (c) mental computation, or (d) estimation. After tallying your results, discuss their implication for elementary mathematics instruction.

2. Miss Brill was discussing rounding with her class, when LeMar made an interesting conjecture. “I think,” the boy announced,
“that when the next digit to the right is five, you should always round up.” Asked to justify his conjecture, LeMar went to the board and wrote the following:

<table>
<thead>
<tr>
<th>Keep the digit when the next digit is</th>
<th>Round up when the next digit is</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

"Note," he instructed, "that there are five entries in the keep-the-digit list, and only four in the round-up list. For example, 3.0, 3.1, 3.2, 3.3 and 3.4 all are rounded down to 3, and only 3.6, 3.7, 3.8, and 3.9 are rounded up to 4.0. This is not fair. Therefore, five should go with the round-up group. For example, 3.5 should be rounded up to 4.0.”

Evaluate LeMar’s argument.

3. Mrs. Xu teaches kindergarten. Her textbook introduces estimation with activities such as filling a gallon jug half way up with marbles, asking students to write down their guesses, and then checking to see who is the closest by counting the marbles. After this, another jug filled three-fifths of capacity is presented and the process is repeated. In turn, jugs, four-fifths and completely filled are presented.

a. Does it make sense to introduce quantity estimation in kindergarten? Why or why not?

b. Would the activity described be an appropriate one for introducing estimation? Why or why not?

c. How might Mrs. Xu modify or supplement the lesson to make her class’ estimation experience more profitable?

4. Professional teachers use curricula material thoughtfully. They use those aspects that help them achieve their aims, modify other aspects to do so, and disregard those aspects that do not effectively help achieve their aims. Below is a brief description of the estimation instruction suggested in Mathematics Their Way (Baratta-Lorton, 1976). Evaluate each aspect described. Indicate whether or not it is consistent with the guidelines recommended in the Student Guide.

a. Mathematics Their Way would have a kindergarten teacher introduce number and measurement estimation.

b. It does not provide explicit guidelines about what size collections should be used but, from its examples (e.g., a larger bag of peanuts, 39 bottle caps, 80 blocks), implies estimation should start with collections larger than about 25.

c. Children are encouraged to estimate the number of items in a jar. Afterward, they are encouraged to evaluate their estimates by counting the number of items in the jar.

d. Children then are given a jar with a larger number of items in it and are encouraged to use their previous estimate.

5. A teacher was conducting a Mathematics Their Way lesson. She showed the class 36 stamps clumped together in a clear sandwich bag. After making estimates, the children counted the stamps in the bag. This process was repeated with 55 stamps. The teacher then showed the group another bag with 68 stamps. At that moment a mathematics-education professor entered the room. He looked at the bag, but because the objects were clumped together, the professor did not have a good sense for how many stamps were in it. The children made estimates in the range of 60 to 70.¹ Much to the surprise of the professor, the children were very close. Given their performance on this task, he was flabbergasted that these same children had difficulty gauging whether a collection was bigger or smaller than a benchmark, even when the benchmark was 10 or 5. For example, many of the children indicated that a collection of 8 was not as big as 5 or was bigger than a collection of 10.* What might account for the discrepancy in the children’s performance?

6. (a) Describe an everyday situation where a

* This is based on data gathered on potentially gifted kindergartners who had estimation instruction as outlined in Mathematics Their Way (Baroody & Gatzke, 1991). Many appeared to have an overexaggerated mental image of 10, identifying collections of 15 as smaller than 10. Some even seemed to have an overexaggerated mental image of five, identifying collections of 8 as smaller than five.
front-end estimation strategy would be ideally suited. (b) Illustrate this technique with a specific example that could help second graders understand how to use this estimation strategy. Include in your illustration your estimate and how you arrived at it. (Note that first considering Question 6 on page 186 of this guide may help you avoid common pitfalls in answering these questions.)

7. Preservice teachers were asked to come up with a problem that could be used to introduce content covered in this chapter. One group came up with the following word problem:

- **Teri’s Blouses (♦ 2-3).** Teri bought three blouses costing $4.95, $6.94, and $7.12. Use a front-end strategy to estimate how much Teri spent.

Evaluate the value of this word problem as a vehicle for introducing a front-end estimation strategy using the investigative approach.

8. Asked how much $42 + 15$ is, Melvin—a third grader—counted on (correctly but laboriously): “$42$ (pause), $43, 44, 45 \ldots 57$.” Even for the simpler problem $40 + 10$, Melvin counted to determine the sum. For a problem $30 + 30$ or $50 + 40$, Melvin did not even try to compute but simply guessed. (a) In what way is Melvin’s mental computation incomplete? (b) On what should instruction focus? (c) To be successful, what prerequisite skills should Melvin’s teacher check? That is, what counting and arithmetic skills provide a basis for furthering Melvin’s mental calculating?

9. (a) Is there any advantage to mentally adding from left to right rather than from right to left (using the standard renaming algorithm)? Briefly justify your answers. (b) Briefly explain how the game Grid Race (see Activity File 7.4 on page 7-28 of the Student Guide) can help children devise a partial-decomposition mental-calculation strategy.

**PROBLEMS**

- **A Hefty Hike (♦ 3-8)**

About how many days would it take the average student in your class to hike from New York to San Francisco?

- **Word Limit (♦ 3-8)**

School and journal assignments sometimes specify a word limit. Estimate the number of words in Unit 7•1 of this chapter. Explain how you arrived at your estimate. Briefly justify your strategy.

- **Alphabet Addition ‡ (♦ 5-8)**

In the alphabet code below, two wrongs do make a right. Assign a different digit to each different letter so that the expression below works out. The letter $O$ represents zero. Hint: Figure out the value of $G$ first.

$\begin{array}{c}
\text{W} \\
\text{R} \\
\text{O} \\
\text{N} \\
\hline
\text{G} \\
\text{R} \\
\text{I} \\
\text{G} \\
\text{H} \\
\text{T}
\end{array}$

- **More Alphabet Addition (♦ 2-5)**

Analyze the alphabet addition expression below and then answer the questions that follow. Note that each different letter represents a different digit.

\[ \begin{array}{c}
\text{N} \\
\text{O} \\
\hline
\text{G} \\
\text{O}
\end{array} \]

(a) Could the sum be odd? Why or why not? (b) Could $O = 0$? Why or why not? (c) Could $O = 5$ or more? Why or why not? (d) Could $N = 4$? Why or why not? (e) What is the maximum value of the sum?

- **Alphabet Subtraction (♦ 3-8)**

If $N = 9$, $R = 3$, and $W = 5$, assign a different digit to each remaining letter in the expression below so that it works out. Hint: Consider first what $P$ might be.

\[ \begin{array}{c}
\text{P} \\
\text{L} \\
\text{A} \\
\hline
\text{T} \\
\text{E}
\end{array} \]

ANSWER KEY for Student Guide

Key for Investigation 7.1 (page 7-2)

Part II: Miscalculating Calculators

(1) 12 was divided by 900. (2) 2,795 was keyed in instead of 27.95. (3) When adding dollars and cents, you can't possibly have a thousandths place. $26.12 was keyed in as 2.612. (4) An item ($199.99) was left out; the total should be $518.94. (5) The total of $478.30 is exactly $100 less than the correct total of $578.30. Apparently $138.85, $145.75, or $156.90 was keyed in incorrectly (without a hundreds digit). (6) A score (90) was keyed in twice; the correct total is 290. (7) The equal sign was hit twice adding a third $9.65; the correct total is $29.77. (This cannot happen with some calculators.) (8) The addends 12, 8, and 15 were added to get a total of 35. Either this partial sum was then multiplied by the last number 5 or 15 was keyed in incorrectly as 150 and added to the other three numbers.

Key for Investigation 7.3 (page 7-9)

Task II

To have a 4-digit sum, F must = 1, and so O cannot also equal 1 and T must be a digit 5 to 9. However, if T = 5, then O can only = 0. However, 0 + 0 = 0, and so the digit in the ones place of the sum would have to be O, not R). If T = 6, then O could equal 2 (6 + 6 = 12) or 3 (if W + W > 9). If O = 2 and O + O = R, then R = 4. This leaves 3, 5, 7, 8, and 9 as possible digits for W. However, the last four possibilities can be eliminated because they would have sums greater than 9 and we are assuming O = 2, not 3. Is it possible that O = 3, not 2? No. If O = 3, then both T and R would equal 6 because O + O = R. Other possible solutions exist.

Task III

(a) Both expressions have more than one solution. (b) The sums could not be odd. In both, the ones digits are the same and the sum of two odd numbers or two even numbers is even. (c) In both, the ones digit could not be 0. (The ones-digit sum is represented by a different letter.) (d) For ME + ME = US, the smallest sum possible is 13 + 13 = 26, and the largest sum possible is 48 + 48 = 96. For NO + GO = NIX, N + G must result in a carry. Hence, N must equal 1. If N = 1, then G must be 8 or 9. If G = 8, then O must be 5 or larger. O cannot be 5 because 15 + 85 = 100. Thus 16 + 86 is the lowest sum. If G = 9, then O cannot be 5 or greater (e.g., 18 + 98 = 116, but this is not possible because N ≠ 1). With G = 9, the largest sum possible is 14 + 94 = 108. With G = 8, the largest sum possible is also 108 (89 + 89 = 108).

Key for Investigation 7.4 (pages 7-10 and 7-11)

Task I

d. One approach is to simplify the expression

\[
\frac{322,385}{4,865}
\]

and use compatible numbers (e.g., 300 ÷ 5 or 320 ÷ 4). This makes it relatively easy to see that the quotient will be a two-digit number (about 60 or 80).

Task II: Example 1

a. Possibly true. For example, although the addends could be 15, 20, and 25, they could also be 35, 30, and 25. The first example is consistent with the conjecture; the second is not.

b. False. If one addend was 70, the sum could not be a number less than 90. For instance, if the other two addends were 10 and 10 (the smallest two-digit numbers possible), the sum would be 90.

c. True. If the first two addends are greater than 35, then their sum is greater than 70. Thus, the third would have to be less than 20 to sum to 89 (e.g., 36 + 36 = 72, and the largest number that can be added to 72 to make a sum less than 90 is 17).

d. Possibly true. For example, 29 + 29 is only 58 and, thus, the third addend could be as large as 31. However, the third does not necessarily have to be greater than 30.

Key for Probe 7.1 (page 7-16)

Part I: The Case of Amanda

1. Amanda had no difficulty rounding 2-digit numbers to the nearest ten or 3-digit numbers to the nearest hundred. Why then did she have difficulty rounding 3-digit numbers to the nearest ten? Unfortunately, the girl's errors were not systematic. However, a key clue to Amanda's difficulty was her inability to construct the appropriate number line. This suggests Amanda probably could not count by
190
tens over 100. That is, she did not realize the repetitive nature of counting by tens.

2. To help remedy Amanda’s difficulty, a teacher could make up cards with the count-by-ten sequence to 1000. Once the count-by-tens sequence to 1000 was secure, the girl could be encouraged to use this knowledge to round 3-digit numbers to nearest tens (e.g., asked: Is 654 closer to 650 or 660?).

Key for Investigation 7.6: Questions for Reflection (page 7-17)

3. Rounding down could be used to indicate the minimum value of a range, and rounding up could be used to indicate the maximum value.

Key for Investigation 7.7 (page 7-19)

Task 1: A Long Season. Students may find the following hints helpful: What portion of a year does a baseball season cover? About how many weeks would this be? About how many games a week do you think a team would play? Answer: Major league teams play a 162-game season.

Task 2: A Big Distance. Students may find the following hints helpful: How many time zones are there and how big is a time zone at the equator? What knowledge might you have that would help you estimate the size of a time zone? The actual distance is about 24,900 miles.

Task 4: Many People. The population of the United States is over 250 million.

Key for Probe 7.2 (page 7-25)

2. To mentally calculate sums and differences, many students use the standard right-to-left (renaming) algorithm. Although this algorithm is an efficient method for performing written calculations, it is a relatively inefficient way to carry out mental calculations (Plunkett, 1979). The carrying or borrowing required by such strategies places a heavy demand on working memory, which increases the chances of errors. Moreover, the partial sums must be recalled in left-to-right order—in the reverse order they were computed. Computing the sum of 157 + 286, for instance, requires the following steps: (a) add the ones digits (7 + 6 = 13); (b) store the partial sum 3; (c) append the 1 to the tens column, (d) add the tens digits (1 + 5 + 8 = 14); (e) store the partial sum 4; (f) append the 1 to the hundreds place; (g) add the hundreds digits (1 + 1 + 2 = 4); (h) store the partial sum 4; (i) state the partial sums in the reverse order computed (four hundred forty-three).

In contrast, a left-to-right decomposition strategy reduces the demands on memory by eliminating the need to carry or borrow and keeping a running total. Such a strategy also allows stating the partial sums of an answer in the order they are produced. For 157 + 286, for instance, a decomposition strategy would involve (a) decomposing the addends (157 = 100 + 50 + 7 and 286 = 200 + 80 + 6); (b) adding the hundreds of each addend (100 + 200 = 300), (c) readjusting the 300 upward to 400 because an inspection of the tens column indicates a carry, and stating straight away the partial sum (“four hundred”); (d) combining the tens (50 + 80 = 130), ignoring the hundreds digit, which has already been taken into account; (e) adjusting the 30 upward to 40 because an examination of the ones column indicates a carry, and stating “forty”; (f) combining the ones (7 + 6 = 13); (g) stating the ones digit “three” because the 10 has already been taken into account. Note that this strategy also has the advantage of using nice numbers—round, easily computed tens, hundreds, and so forth. Because of its relative ease, a left-to-right decomposition strategy is widely used by schooled and unschooled children and adults all over the world (e.g., Ginsburg, Posner, & Russell, 1981).

3. Although easier to use than the standard right-to-left algorithm, the decomposition strategy is still fairly demanding when renaming is involved. This is especially true for subtraction (Wolters, Beishuizer, Broers, & Knoppert, 1990). Because the decomposition strategy can result in many errors once renaming is introduced, a number of European countries (e.g., England, Germany, Netherlands) have started teaching a less error-prone partial decomposition strategy. Unlike the decomposition strategy that involves adding or subtracting two nice numbers, the partial decomposition strategy involves adding or subtracting a nondecomposed (nonnice) number and a decomposed (nice) number. For 157 + 286, for example, 157 is added to 200, the resulting partial sum (357) is then added to 80, and fi-
nally this partial sum (437) is added to 6. The partial decomposition strategy is rarely used by U.S. children, in part, because many have not learned how to add an undecomposed number and a nice number (Fuson, 1990).

**Key for Activity File 7.6 (page 7-30)**

A would have to be 1848 because this is the earliest date and his election as vice president had to occur before he assumed the presidency and sent Perry to Japan. The most reasonable number for B would be 48. Aside from the fact that it is unlikely that a 13- or 21-year-old would be elected to national office, the U.S. Constitution stipulates that the minimum age for the office of vice president must be at least 35. It could not be 74 because this is the most logical choice for his age at death. C must be 1850 because it is between 1848 and 1853. D comes down to a choice between 13 and 21. However, only the 13 fits the ordinal suffix *th*. By the process of elimination, E is 1848 and G is 74. Checking confirms these answers: If he was 48 in 1848, then he must have born in 1800; 21 years after 1853 would be 1874; and 1874 - 1800 is 74.