UNDERSTANDING BASE-TEN, PLACE-VALUE SKILLS: READING, WRITING, ARITHMETIC WITH MULTIDIGIT NUMBERS

TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 6•1: Grouping and Place Value

The key aims of this unit are to help readers explicitly understand the grouping and place-value concepts underlying our written number system, why children often have difficulty with these concepts, and how these concepts can be taught in a purposeful, meaningful, and thought-provoking manner. Investigation 6.1: Analyzing Various Numeration Systems (pages 6-4 to 6-6 in the Student Guide) was designed to help achieve the first aim above. Part II of this investigation can also serve to promote a discussion about why children have difficulty understanding the rationale for multidigit numbers. (They informally view numerals such as 12 exclusively in terms of a unit-based conception—as a collection of 12 items.) Moreover, the investigation can serve to illustrate how analyzing ancient numeration systems (mathematical history) and children's literature can serve as a basis for exploring grouping and place-value ideas in an interesting, understandable, and inquiry-based manner.

Probe 6.1: Analyzing Multidigit Models (pages 6-10 and 6-11 in the Student Guide) and Probe 6.A: Analyzing a Multidigit Math Game (pages 145 and 146 of this guide) were designed to help readers construct an explicit framework for analyzing and judging the value of concrete and pictorial models of multidigit numbers. The hope is that they will recognize that not all models of multidigit numbers are alike—that one model may fit a child's developmental needs well, while another may not. For example, for a child struggling to construct a grouping concept, a model involving interlocking blocks might be helpful, whereas an abacus might be too abstract.

Although some mathematics educators now dismiss the teaching of other bases as outdated—as an unfortunate relic of the New Math (e.g., Fehr, 1966)—our experiences with elementary-level children, preservice teachers, and in-service teachers suggest that it can be invaluable in several ways: (1) Purposeful, meaningful, and inquiry-based learning of other bases can enhance a positive disposition toward mathematics (Baroody, 1987). Our experience is that children find such activities both interesting and challenging. Unfortunately, many pre- and in-service teachers were introduced to other bases in a meaningless manner and were confused by the topic. Unraveling the mysteries of other bases can boost their sense of mathematical power. (2) Learning about other bases can deepen both children's and adults' understanding of grouping and place-value and, thus, our decimal notation system. (3) Learning about other bases can involve students of all ages in the processes of mathematical inquiry, such as (inductive and deductive) reasoning and testing or evaluating conjectures. (4) Knowledge about other bases can promote thinking mathematically—analyzing everyday situations in terms of mathematical ideas such as grouping.

Investigation 6.2: Thinking About Other Bases (pages 6-12 and 6-13 of the Student Guide), Investigation 6.A: Base Games (pages 147 and 148 of this guide), Investigation 6.B: An Interesting Conjecture About Converting to Other Bases (page 149 of this guide), and Investigation 6.C: Applications of Other Base Systems (pages 150 and 151 of this guide) were designed to achieve one or more of the objectives outlined above.

Unit 6•2: Operations on Multidigit Numbers

The aim of this unit is to help readers appreciate the importance of encouraging children to invent their own informal models of multidigit arithmetic and using these concrete models as a basis of devising their own symbolic procedures or reinventing standard procedures. Probe 6.2: Pedagogical Questions About Using Manipulatives to Teach Multidigit Addition and Subtraction (page 6-16 of the Student Guide), Investigation
6.3: Renaming with Egyptian Hieroglyphics (page 6-17 of the Student Guide), Investigation 6.5: Symbolic Multidigit Multiplication (pages 6-22 and 6-23), and Probe 6.3: Reinventing Multidigit Division Algorithms (pages 6-25 and 6-26 of the Student Guide) were designed to help underscore these aims.

Furthermore, Investigation 6.4: Largest (Smallest) Sum (Difference) (page 6-21 in the Student Guide) illustrates how practice of multidigit-arithmetic skills can be done in a purposeful and entertaining manner. Probe 6.B: Some Alternative Algorithms for Multidigit Addition and Subtraction (page 152 of this guide) and Investigation 6.D: Other Multidigit Multiplication Procedures (pages 153 and 154 of this guide) can serve to underscore the point that there are various algorithms for operating on multidigit numbers, some of which are at least as efficient as the standard algorithms traditionally taught in school.

SAMPLE LESSON PLANS

Project-Based Approach

Alternative 1. Using SUGGESTED ACTIVITIES 1 to 9 and 11 (on pages 155 to 157 of this guide) as a menu, have small groups of students choose a project. Particularly if different projects are chosen, have each group share the results of its project with the rest of the class. Most students find Part IV (Mayan Number Symbols) particularly challenging. Frequently, someone will conjecture that the means times 20. This is a reasonable conjecture given that means one and represents 20. Encourage students to test this conjecture. They should quickly discover counterexamples. For example, the conjecture suggests that 21 should be or , but it is, in fact . If students need a hint, encourage them to consider 421 and 400. With any luck, someone will recognize that the Mayan numbers are a base-twenty, place-value system.

Alternative 2. An instructor may wish to choose one of the problems on pages 162 and 163 of this guide to serve as a worthwhile task. The last seven problems, in particular, should be challenging and provide a basis for exploring grouping and place-value issues.

Alternative 3. An instructor could model the investigative approach on multidigit subtraction by, for example, using Activity File 6.4: A Unit on Temperature Drops (on page 6-18 of the Student Guide) as a lesson plan and having the class read the children’s books and carry out the science experiment described in this activity file.

Single-Activity Approach

Lesson 1. Many adult students have only a fuzzy understanding of what is meant by grouping and place value and others may not understand the distinction between these terms at all. To explicitly define these terms and distinguish between them, have students read the introduction and complete Part I of Investigation 6.1: Analyzing Various Numeration Systems (on pages 6-4 and 6-5 of the Student Guide).

To illustrate children’s difficulties with writing multidigit numerals and operations on multidigit numbers, have them present numeral-writing or multidigit arithmetic tasks to first- and second-graders. Alternatively, show them videotapes of interviews (e.g., Segment 5 of “Grade 1: Connecting Informal and Formal Mathematics” and Segments 3 to 6 of “Grade 2: Place Value”—both of which are from Children’s Mathematical Thinking: Videotape Workshops for Educators by H. P. Ginsburg and colleagues, © 1992 by the Everyday Learning Corporation). To further illustrate children’s incomplete knowledge of grouping and place-value ideas, have students work on and discuss Part II of Investigation 6.1 on page 6-5 of the Student Guide. For example, writing 12 in Egyptian hieroglyphics as |||||||||||| suggests that a child views multidigit numbers in terms of a unit-base concept of number, not in terms of base-ten, place-value concepts. Part II can also serve to spark a discussion of how the investigative approach can help children construct these concepts. Parts III, IV, and V (page 6-6 of the Student Guide) can further all the previously discussed aims.

Most students find Part IV (Mayan Number Symbols) particularly challenging. Frequently, someone will conjecture that the means times 20. This is a reasonable conjecture given that means one and represents 20. Encourage students to test this conjecture. They should quickly discover counterexamples. For example, the conjecture suggests that 21 should be or , but it is, in fact . If students need a hint, encourage them to consider 421 and 400. With any luck, someone will recognize that the Mayan numbers are a base-twenty, place-value system.

Lesson 2. To underscore the value of purposeful (e.g., game-based) and meaningful (e.g., manipulative-based) instruction, have students analyze, for instance, Forward Bowling described in Activity File 6.A on page 145 of Probe 6.A: Analyzing a Multidigit Math Game in this guide. As an extension, ask students to consider how they could use other math games to help children...
learn to read and write multidigit numerals and to perform subtraction on multidigit numbers in base-ten or another base, such as base 4 (Zork).

Multiple-Activities Approach

Lesson 1. For a relatively broad introduction to grouping and place-value instruction, an instructor might choose the following sequence of activities:

1. To define and compare the terms base-ten and place value explicitly, an instructor can have students begin with the activities outlined in the "bulletin board" on page 6-1 of the Student Guide or do selected portions of Investigation 6.1: Analyzing Various Numeration Systems (pages 6-4 to 6-6 of the Student Guide). These activities can also serve to prompt a discussion of children’s difficulties (e.g., viewing a multidigit number such as 12 in terms of a unit-based concept—as 12 things—instead of grouping and place-value concepts—12 = one group of 10 and two singles) and how an investigative approach can help children construct these concepts.

2. Discussing Probe 6.1: Analyzing Multidigit Models (pages 6-10 and 6-11 of the Student Guide) can help students construct an explicit framework for matching concrete models of grouping and place-value to a child’s developmental level. It should also help them see that not all concrete models accomplish the same thing. For example, although all the models described in this probe involve grouping, some (e.g., Model A) are relatively concrete, while others are fairly abstract (Model D), and only some models involve place value (e.g., Model D), ii: the peg and disk of the same color, iii: colored chips and trading board, and iv: the abacus).

3. Engaging students in, for example, Activity III of Investigation 6.A: Base Games (page 148 of this guide), Parts I, II, and/or III of Investigation 6.B: Thinking About Other Bases (pages 6-12 and 6-13 of the Student Guide, and Part I of Investigation 6.C: Applications of Other Base Systems (page 150 of this guide) can help them overcome discomfort with the other bases, deepen their understanding of grouping and place value, and engage them in critical thinking. For example, Question 1 in Part I of Investigation 6.2 inevitably raises the question: How can ten items be represented in base 12? Through reflection and group or class discussion, students should recognize that 10$_{12}$ is not appropriate because this numeral represents one group of 12 and no singles in base 12. Perhaps with help, students will recognize that some other arbitrary symbols can be used to represent ten items in base 12. Likewise, they should see that 11 cannot represent eleven items in base 12. An instructor may wish to point out that one convention for handling such situations is to use the letters of the alphabet (a sequence of symbols for which the order is known to almost all school-age children). For example, in base 12, A can represent 10, and B can represent 11. In brief, students may recognize, perhaps for the first time, that 10 in our decimal system stands for one group of ten and no singles and that the symbol 10 represents the number that names a particular base (e.g., in base four, 10 represents one group of four). An instructor may wish to underscore that calling 10 ten in any base but base 10 can be confusing to children and, thus, should be avoided. Ten is the base-ten name for a group of 10 items. In the base-four language of the fictitious Zorkians, 10 items would be called two zork and two (two groups of four and two singles.)

Lesson 2. For a relatively broad introduction to written procedures for operating on multidigit numbers, an instructor could choose, for instance, the following sequence of activities:

1. To help adult students consider how a teacher could use the investigative approach to prompt children to invent concrete and then written procedures for adding or subtracting multidigit numbers, an instructor might have them consider Question 1 of the Conceptual Phase and Questions 1 and 4 of the Connecting Phase in Probe 6.2: Pedagogical Questions About Using Manipulatives to Teach Multidigit Addition and Subtraction (page 6-16 of the Student Guide) and Investigation 6.3: Renaming with Egyptian Hieroglyphics (page 6-17 of the Student Guide). Examining Boxes 6.2 (An Example of the Investigative Approach for Introducing the Standard Addition Algorithm) and 6.3 (Creating Cognitive Conflict to Encourage Self-Correction of Faulty Procedures) on page 6-20 of the Student Guide can illustrate how a teacher can promote learning indirectly by prompting reflection.

2. To illustrate how computational practice can be entertaining and educational, an instructor could have the class play the game described in Investigation 6.4 Largest (Smallest) Sum (Differ-
ence) on page 6-21 of the Student Guide. In addition to being enjoyable, this game can foster a number (operation) sense and also raise interesting questions about probability (see, e.g., Problem 2 in Investigation 13.8: More ’At Least’ Problems on page 13-25 of the Student Guide). Begin with the relatively simple game Largest Sum. Next, try Smallest Sum, a procedure that will require players to radically rethink their strategies. Move on to Smallest Sum. Because answers can now be less than 0, give students a choice as to which version of the game they want to play: (a) Whole-Number Version, where the whole number closest to 0 wins; (b) Integer Version, where the smallest integer (largest negative number) wins; or (c) Absolute-Value Version, where the integer (positive or negative) closest to 0 wins. Finally, try Largest Difference.

3. To provide an overview of multidigit multiplication and division, complete as much of Investigation 6.5: Symbolic Multidigit Multiplication (pages 6-22 and 6-23 of the Student Guide) and Probe 6.3: Reinventing Multidigit Division Algorithms (pages 6-25 and 6-26 of the Student Guide) as time permits.

SAMPLE HOMEWORK ASSIGNMENTS

Lesson 1

Read: Unit 6•1 of chapter 6 in the Student Guide.

Study Group:

• Questions to Check Understanding: 1a to 1g, 2, 11 and 12 (pages 157 to 160).

• Problem: The Meter is Ticking (page 163).

• Bonus Problem: Conflict Over the Lap Counter (page 163).

Lesson 2

Read: Unit 6•2 of chapter 6 in the Student Guide.

Study Group:

• Questions to Check Understanding: 1h to 1k, 16, 17, 18, 19 (pages 158 to 161).

• Investigation 6.D: Other Multidigit Multiplication Procedures (see pages 153 and 154).

• Bonus Problem: Chip-Trading Problems (page 163).

Individual Journals: Writing or Journal Assignments 1 and 3 (page 161).

FOR FURTHER EXPLORATION

ADDITIONAL READER INQUIRIES

Probe 6.A (pages 145 and 146)

Analyzing a Multidigit Math Game illustrates how keeping score of a game can provide a purposeful and meaningful basis for constructing multidigit numeration concepts and skills.

Investigation 6.A (pages 147 and 148)

Base Games illustrates entertaining ways to promote learning about grouping by numbers other than 10 and applying place-value concepts to other base systems.

Investigation 6.B (page 149)

An Interesting Conjecture About Converting to Other Bases illustrates how work with other bases can involve children in the processes of mathematical inquiry.

Investigation 6.C (pages 150 and 151)

Applications of Other Base Systems illustrates how grouping by numbers other than 10 plays an important role in everyday life and includes an application to computer science. This, like Investigations 6.A and 6.B, may help to convince readers that learning about other bases is worthwhile.

Probe 6.B (page 152)

Some Alternative Algorithms for Multidigit Addition and Subtraction illustrates that there are various "correct" procedures for performing multidigit addition and subtraction.

Investigation 6.D (pages 153 and 154)

Other Multidigit Multiplication Procedures further illustrates that there is not one correct method for calculating an answer. (Text continued on page 155.)
Probe 6.A: Analyzing a Multidigit Math Game

The aim of this probe is to help you see how valuable the everyday activity of keeping score for a game can be in constructing multidigit numeration concepts and skills. It also serves to review and tie together the material in chapter 6 of the Student Guide.

In the Wynroth (1986) Math Program, students are introduced to multidigit concepts and skills concretely with games such as Forward Bowling described in Activity File 6.A below. These games and, thus, multidigit numeration and its underlying rationale are introduced late in kindergarten or early first grade.

Activity File 6.A: Forward Bowling

- Multidigit numeration + base-ten, place-value concepts
- K-2
- Two to four children or two small teams of two or three children

Figure A below illustrates that a player’s (team’s) score at the start of a turn is 96. Note that the multidigit numeral is connected to a concrete model of the number. On their turn, players knock down bowling pins with a bean bag and record their score by stacking interlocking cubes in the ones column (see Figure B below). The rules specify that only nine cubes may remain in a column. If there are more, children are instructed to count (the top) 10 cubes and trade them in for a cube that is placed in the next position to the left (see Figure C below). After creating a concrete embodiment of the score, a child then labels the columns with numerals and reads the score (see Figure D below). The winner is the player with the biggest score after a preset time or the first to achieve a preset score such as 199.

A. Score at start of a turn

B. Scored 7 more points on the turn

C. Trade Up

D. Complete any additional trades required and change the numeral designation
Questions Regarding Forward Bowling

1. How can the scoring procedure for Forward Bowling, illustrated on the previous page, help children learn to read and write multidigit numbers in a meaningful fashion?

2. Which of the following base-ten, place-value concepts or skills does this model involve? Briefly justify your answer.
   a. recognition of ones, tens, and hundreds places (e.g., for the numeral 135, the 5 is the ones place, the 3 is in the tens place, and the 1 is in the hundreds place)
   b. position defines the value of a digit (e.g., for the numeral 135, the 3 in the tens place is a multiple of 10—specifically, three tens or 30)
   c. zero as a place holder (e.g., for the numeral 206, the 0 indicates no groups of ten)
   d. digit values equal the whole (e.g., the value of the individual digits in 162 must sum to 162, that is, 100 + 60 + 2 = 162)
   e. 10-for-1 trading between successive places (e.g., 10 ones can be traded for a ten and vice versa)
   f. base-ten equivalents (e.g., 10 ones = 1 ten and 10 tens = 1 hundred)
   g. informal basis for positional notational with renaming (e.g., 3 tens + 15 ones = 4 tens + 5 ones)
   h. explicit knowledge of positional notational with renaming (e.g., 3 tens + 12 ones = 42 and vice versa)

3. The scoring procedure illustrated in Activity File 6.A on the previous page of this probe is an example of which model depicted on page 6-10 of the Student Guide? To decide, consider the following questions:
   a. Is it a proportional model? Why or why not?
   b. Does it involve (i) physically grouping 10 items to form a single group of ten, (ii) concrete 10-for-1 trading (trading 10 items for a single item that physically resembles the 10 items), or (iii) abstract 10-for-1 trading (trading 10 items for a single item that does not physically resemble the 10 items)? Why?
   c. Does it involve place value?

4. For a child struggling to understand a grouping concept, does this model provide the most concrete embodiment of this concept?

5. The model provides a concrete embodiment for what instruction later? How so?

*Note that recognition of place names (Item a) can be learned by rote. Generally speaking, Items b and c require a conceptual understanding. Items d to h represent aspects of a relatively deep understanding of place value.
Investigation 6.A: Base Games*

♦ Grouping and place-value concepts ♦ K-8 ♦ Any number

The aim of the following activities is to help students construct general grouping and place-value concepts, which can deepen their understanding of decimal numeration. It is especially important for those who have had little or no meaningful instruction on other base systems to work through the activities with manipulatives or drawings of manipulatives. To deepen your understanding of other bases and grouping and place-value concepts, try the activities with your group or class.

**Activity I: A Review of Zork**

Miss Brill used a free period to visit Ms. Wise's classroom to see what she could learn. Ms. Wise, in the guise of Zork, had previously introduced the class to grouping by four with base-four blocks (see pages 23 to 25 of Probe 1.C in this guide).

1. Ms. Socrates asked if anyone was going to have a birthday soon. Tonja raised her hand and said she would be 11-years-old the next day. Ms. Wise then asked a strange question, "On the planet Zork, how old would Tonja be on her next birthday?" (a) How would 11 be represented with base-four blocks? (b) How would Zorkians record Tonja's age?

2. Ms. Wise then noted that her cat had just turned 8-years-old. (a) How would 8 be represented with base-four blocks? (b) How would Zorkians record 8? (c) Mitchell suggested 2. What important point about numeration systems does this provide an opportunity to discuss?

3. Next, Ms. Wise asked Miss Brill's age. Miss Brill indicated she was 23. How would 23 be represented with (a) base-four blocks and (b) Zorkian numbers?

4. Lastly, Ms. Wise noted, with a twinkle in her eye, that she was about 35-years-old. How would this be represented with base-four blocks and Zorkian numbers?

**Activity II: The Block-Trading Game With Red and Blue Blocks**

Ms. Wise asked, "In our everyday number system, by what number do we group? That is, how many ones must be collected to trade for the next larger unit? How many of that unit must we collect to trade for the next larger unit and so forth?" The class agreed it took 10 ones to trade for a ten, and 10 tens to trade for a hundred, and 10 hundreds to trade for a thousand. "Because we group by tens," noted Ms. Wise, "we call our number system a base-ten system."

Ms. Wise then posed the following series of review questions, "With Zork blocks, how many cubes must you collect to trade for a zork (a long)? How many zorks must you collect to trade for a super zork (a flat)? How many super zorks must you collect to trade for a super-duper zork (a large cube)?" After the class concluded that the Zork number system was a base-four system because it grouped by four, Ms. Wise gave the class two different groups of blocks.

1. The blue blocks consisted of:

   - cube
   - long
   - flat
   - large cube

   Ms. Wise asked, "These blocks are grouped by what number? Why?" What answer and explanation did she expect?

*Multibase blocks can be ordered from: Tools for Teaching
Bear Lake Village, 3030 E. Semoran Blvd.
Apopka, FL 32703
Phone: 407-682-6626; Fax: 407-880-8084
Investigation 6.A continued

2. The red blocks consisted of:

- cube
- long
- flat
- large cube

Ms. Wise asked what base system these blocks represented and asked her students to justify their conclusion. What answer and justification did she expect?

3. (a) Ms. Wise had her class play the Block-Trading Game (see page 6-8 of the Student Guide) with blue blocks. If you rolled two regular dice and got a total of seven, what would be the most economical way to collect blue blocks to represent your score? (b) If the game is played with red blocks, what would be most economical way to collect red blocks to represent this score? (c) How might this help children construct a part-whole concept?

4. The winner of the Block-Trading Game is the first player to collect a large cube. What is a large blue cube worth? What is a large red cube worth? (Express your answers as a base-ten number.)

Activity III: Gauge My Age II

This activity, which builds on children’s natural interest in people’s ages, provides a meaningful way to practice writing numbers in other bases. Translating an age such as 11 years (base 10) into other bases can be done with aid of manipulatives or, with more advanced students, mentally. Ms. Wise, for example, had her class determine ages in base-four (see Activity I on the previous page) and then expanded her chalkboard chart to include columns for base 5 and base 6. Complete the chart below:

<table>
<thead>
<tr>
<th>Age our system (Base 10)</th>
<th>Age on Zork (Base 4)</th>
<th>Age blue blocks (Base 5)</th>
<th>Age red blocks (Base 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>My cat</td>
<td></td>
<td>204</td>
<td></td>
</tr>
<tr>
<td>Tonia</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Miss Brill</td>
<td>23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ms. Socrates</td>
<td>35</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Questions for Reflection

Miss Brill really liked Ms. Wise’s lesson on base systems. She tried the lesson with her class but quickly ran into trouble. A portion of her age chart is shown to the right. Rodney put into words the question on many children’s minds, “If Andy’s sister is really 4-years-old, how can she be ten? And if Andy is really 10-years-old, how can he be twenty-two? Does being on Zork age you?”

1. Miss Brill and her class called 104 ten. Is this an accurate name for 104? Why or why not?

2. What should 104 be called on Zork? How could earthlings unfamiliar with the Zork counting system refer to 104?

3. What would 224 be called on Zork? How could earthlings unfamiliar with Zork refer to 224?
Investigation 6.B: An Interesting Conjecture About Converting to Other Bases

- Problem solving + looking for patterns + making and testing a conjecture + logical reasoning - Any number

This investigation illustrates a case where an exploration of other bases lead to an interesting conjecture. It asks students to use their understanding of base systems to evaluate the conjecture. To test your own understanding, try the investigation yourself. Discuss your conclusions with your group or class.

While playing *Gauge My Age II* (Activity III of Investigation 6.A on the previous page), a teacher asked his class to translate his age of 47 into Zork. Most students arrived at the answer of 233₄. Alison disagreed, claiming the answer was 113₄. Asked how she arrived at the answer, the sixth-grader explained, "I used a calculator. To find the number of groups of four, I divided the age by four." (She used the \(\text{INT ÷}\) key on a Texas Instruments Math Explorer. Because the integer division function indicated that the quotient of 47 ÷ 4 was 11 with a remainder of 3, Alison concluded the Zorkian equivalent of 47 had to be 113₄.

Asked to justify her procedure, Alison said, "I tried it with the other ages and it worked." The class had previously translated the ages of 7, 11, and 14 into Zorkian numbers. Alison's procedure and previous solutions are listed below:

\[
\begin{align*}
7 & \text{ INT ÷ 4 = 1 r 3 13₄} \\
11 & \text{ INT ÷ 4 = 2 r 3 23₄} \\
14 & \text{ INT ÷ 4 = 3 r 2 32₄}
\end{align*}
\]

Impressed by Alison's strategy, the teacher asked the class what they thought.

"It [113₄] can't be right," responded Alexi. He noted that two sixteens, three fours, and three ones made 47, but one sixteen, one four, and three ones did not.

Arianne justified her answer of 233₄ with base-four blocks.

Alison was not convinced, and so the teacher asked, "Can both 233₄ and 113₄ be correct?"

Alison rejoined, "I can prove my answer is correct. She proceeded to make the drawing below and count the number of groups of four. "There are 11 and 3 leftover."

Examine the diagram carefully, Alexi, observed, "Thanks Alison, this diagram proves our case."

1. Why did Alison's calculator-based method work for 7, 11, and 14 but not 47?
2. How could Alison's drawing above be used to prove Alexi's solution of 233₄?
Although we often group by tens in everyday life, there are many occasions where we use a different grouping system (e.g., see Activity File 6.B below). The aim of this investigation is to help make students aware of some of these everyday applications of other base systems. To see what is involved and to perhaps extend your own understanding of grouping, answer the questions below.

Activity File 6.B: Relating Other Bases to Household Situations

This activity can help children see that you can group by numbers other than 10, as well as learn common measurement equivalents. A teacher may wish to begin with a problem such as:

**A Soda Order** (3-6). The picnic-planning committee agreed that 2 glasses of soda would be a good portion for each of the children and adults attending the grade-level picnic. They asked their teacher how much soda they should order for 112 people attending the picnic. "I don’t know," responded Ms. Ramas, but there are cup, pint, and quart containers in the cabinet in back.

If 2 glasses is 2 1/2 cups, how many quarts of soda does the planning committee have to buy?

By using a smaller container to fill a larger one, children should discover that 2 cups = 1 pint and 2 pints = 1 quart. If each of 112 people gets 2 glasses, which equal 2 1/2 cups, then 280 cups are needed. The number of pints needed would be 280 ÷ 2 or 140; the number of quarts needed, 140 ÷ 2 or 70.

**Soda Special.** The supermarket had a special on gallon containers of soda. How many gallons of soda would be needed for the grade-level picnic?

By experimenting with a quart and a gallon container, students should recognize that it takes 4 quarts to make a gallon. Thus, 70 ÷ 4 or 17.5 gallons would be needed for the grade-level picnic.

**Part I: Everyday Uses**

1. **Common Units.** Some items such as eggs are packaged by the dozen. Twelve dozen (i.e., 144 eggs) is 1 gross, and 12 gross is 1 great gross. (a) What base system does this represent? (b) What other items are packaged according to this system?

2. **Time Measurement.** In measuring time, 60 seconds equals 1 minute and 60 minutes equals 1 hour. (a) What base system is this? (b) Is time measured completely in terms of this system? Consider, for example, a millisecond.

3. **Circular Measurement.** (a) Astronomers, mapmakers, and anyone who must navigate long distances such as sailors and pilots use a system of circular measurements: 60 seconds (60") = 1 minute (1'); 60 minutes (60') = 1 degree (1°). (a) What base system is circular measurement based on? (b) 360 degrees (360°) equal 1 circumference. Is this consistent with this base system?

4. **Money.** Our money system is a conglomeration of grouping systems. Consider the U.S. coins: pennies, nickels, dimes, quarters, half-dollars, and (silver) dollars. (a) Which set of coins is consistent with a base-ten system? (b) Which set of coins is consistent with some other system? What base is it? (c) Which coins are based on a mix of the two base systems?

5. **Dry Measure.** Dry measures are used for produce such as apples and strawberries. Two pints equal a quart, 8 quarts equal a peck, 4 pecks equal a bushel. (a) What grouping systems do dry measures involve? (b) A bushel contains how many pints? (c) If dry measures were entirely a base-two system, a peck would contain how many quarts? (d) A bushel would contain how many pints?

**Part II: Computer Science**

Because it is an electrical device, a computer “thinks” in terms of base two. A circuit can either be off, which can represent 0, or on, which can represent 1. One (1₂) would be represented by...
Investigation 6.C continued

having the first circuit on; two \((10_2)\), by having only the second circuit on (represented by the zig-zag line in Figure A below); four \((100_2)\), by having only the third circuit on, etc.

A.

<table>
<thead>
<tr>
<th>8th</th>
<th>7th</th>
<th>6th</th>
<th>5th</th>
<th>4th</th>
<th>3rd</th>
<th>2nd</th>
<th>1st circuit</th>
</tr>
</thead>
</table>

1. What is the base-two numeral for each of the following base-ten numbers and how would a computer represent each (i.e., which circuit[s] would be on and which would be off in each case): (a) 5, (b) 8, (c) 9, (d) 10, and (e) 16.

2. In Figure B below, show how a computer can represent the base-ten number 22. Indicate which circuits would be on by drawing in a zig-zag line.

B.

3. Figure C below illustrates how the computer adds 1 + 1. Note that the first addend \((1_2)\) is represented by turning on the first circuit. The second addend \((1_2)\) is represented by sending an additional charge to the first circuit. As the first circuit is already charged, the additional charge causes it to overflow (to turn off and to turn on the second circuit), which is equivalent to a sum of \(10_2\) or 4. In Figure D below, illustrate how a computer adds 3 + 2.

C.

initial state: \[\quad\]

added amount: \[\quad\]

sum: \[\quad\]

4. Now consider 3 + 3. In Figure E below, indicate how the computer represents the first addend 3. In order to represent the second addend, which circuits would get a charge (if not already on) or an additional charge (if already on)? What do you suppose happens when an overflow encounters a circuit that is already experiencing an overflow?

5. In Figure F, illustrate how the computer would add the base-ten numbers 15 and 10.

D.

initial state: \[\quad\]

added amount: \[\quad\]

sum: \[\quad\]

E.

initial state: \[\quad\]

added amount: \[\quad\]

sum: \[\quad\]

F.

initial state: \[\quad\]

added amount: \[\quad\]

sum: \[\quad\]

The aim of this probe is to underscore that there are many multidigit addition and subtraction procedures. After reading about the seven methods described below, answer the questions at the bottom of the page.

A. Expanded Algorithm. This method underscores the place value of each partial sum. Summing the partial sums (15 + 90 + 400) can also be done without using a renaming procedure. One way would be to add 400 + 90 first and then count 15 more, or add 490 + 10 + 5. Another way would be to repeat the process illustrated in Steps 1 to 3 until renaming was not necessary.

B. Making-Change Technique. The following mental algorithm was used to determine a customer's change before the advent of the automatic cash registers. The technique is still useful in those situations where an automatic cash register is not available (e.g., garage sales). For 500 - 368, for example, bring the 368 up to a multiple of ten by adding 2 (368 + 2 = 370). Now 370 to 400 is another 30, so 368 to 400 is 32. And 400 to 500 is another 100, so 368 to 500 is 132.

C. Expanded Notation. Note that this method underscores the place value of each digit of an addend.

D. Equal Additions. In effect, 10 is added to 205 and, to keep the difference equal, 10 is also added to 68.

E. Shortcut for Equal Additions. This method transforms an expression into one that is relatively easy to subtract by adding an equal amount to each addend.

F. Korean Method of Subtraction. A ten is renamed as 10 ones. Eight is then subtracted from the 10 (10 - 8) and the remainder (2) is added to the 4 (4 + 2) to make 6.

G. Convenient Numbers. This procedure transforms numbers that are hard to operate on into numbers that are easy to manipulate (e.g., 306 + 199 = 306 + [200 - 1] = [306 + 200] - 1 = 506 - 1 = 505).

Questions for Reflection

1. In Method A, how could children complete Step 4 if they did not know the renaming procedure?

2. Could Methods A and C be used with multidigit subtraction to avoid renaming? Why or why not?

3. Complete the subtraction of 205 - 68 using Method D.
Method I: Napier’s Bones

1. John Napier (1550-1617) invented an interesting calculating device for determining products (see figure below). The rods were made of horn or ivory and hence became known as Napier’s bones. Examine the 2 bone (the first bone on the left in the figure below). What patterns do you notice about it? Do the same for the 5 and the 8 bones.

2. To multiply the three-digit number 258 by 4 (4 x 258):
   (a) lay out the 2, 5, and 8 rods in the order shown above;
   (b) count down to the fourth box of each rod;
   (c) add the numbers in the same diagonal strip (2 becomes the ones digit; 3 + 0 or 3 becomes the tens digit; 2 + 8 or 10 becomes the hundreds and thousands digit).

   Illustrate how the product of 7 x 258 could be determined using Napier’s bones.

3. Illustrate how the product of 317 x 4 could be determined by using the Napier Bones. Indicate with an arrow the boxes you would read. Color code the diagonals to show which would give you the ones digit, which would give you the tens digits, and so forth.

4. How could Napier’s bones help children to determine the product of one-digit times one-digit expressions?

5. (a) Using 37 x 46 as an example, illustrate how Napier’s bones could be used to compute the product of two two-digit factors. Hint: Consider a groups-of interpretation and how 30 x 46 might be done. (b) Now try 25 x 58. Illustrate the Napier Bones you would need and how you would use them to determine the product.

Teaching Tips. Have students construct their own set of Napier’s Bones. On a piece of oaktag, cardboard, or other stiff paper, have them draw in a 10-by-10 grid with a marker, crayon, ink, or dark pencil. Cut up the grid to create 10 strips—10 bones for the numbers 0 to 9. For each strip have the students draw in the diagonals and label each bone with a different skip count. (All the entries in the 0 bone will be 0.)

Note that this activity involves applying a number of mathematical skills purposefully. It entails reviewing multiples or skip counts of the numbers 0 to 9—something many fourth and fifth-graders can use. It also requires measurement, because each strip (bone) should have the same width and length and each cell within the strips should also have the same dimensions. (If the bones are not constructed in a fairly exact manner, they will be hard or impossible to use.)

Show the class how the bones can be used to determine the product of one-digit times one-digit combinations and one-digit times two-digit combinations. Then—like Question 5 above—challenge them to invent their own procedure for determining the product of two two-digit numbers and other larger expressions. Encourage students to use a groups-of interpretation to estimate the answer and to invent their multidigit procedure with the bones. It may help student to consider relatively simple cases such 10 x 47 or 50 x 47 first.
Investigation 6.D continued

Method II: Hindu Frame

Long before Napier, the Hindu’s used a frame that only required knowledge of the basic multiplication combinations. The method is easier to use than Napier’s bones, particularly with numbers of more than two digits. The method is illustrated using 672 x 347 as an example.

1. Set up a 3 x 3 frame.

2. Find and record the product for each cell.

3. Find the sum for each diagonal, carrying a digit to the next diagonal when necessary.

4. The completed frame:

Illustrate how the product of 685 x 204 can be determined using a Hindu frame.

Method III: Egyptian Method

The ancient Egyptians used a very different method for multiplying than we do (Bunt et al., 1976). The Rhind papyrus illustrated how to multiply 12 x 12 in a manner similar to that shown below in Figure A below.* Figure B and C illustrate how the ancient Egyptians would have multiplied 15 x 32 and 11 x 20, respectively. Can you decipher what their multiplication procedure was? Note that the symbol meant “the result is the following.” In the space labeled D below, illustrate how the ancient Egyptians would have represented 14 x 23.

A. 

B. 

C. 

D.

*Problem 32 of the Rhind papyrus appears as the mirror image of Figure A above and without the horizontal lines. Figure A appears as it does to facilitate discovery of the Egyptian multiplication procedure.
QUESTIONS TO CONSIDER

1. Use the digits 1, 2, 3, and 4 to write as many different 4-digit numbers as you can. Next try using the digits 0, 1, 2, 3 to write as many different 4-digit numbers as possible. Can you write as many as you did with 1, 2, 3 and 4? Why or why not?

2. Miss Brill decided to try a number-comparison game in other bases. She used the following comparisons: (i) \(22_4 \leq 15_6\), (ii) \(33_4 \leq 24_6\), (iii) \(103_4 \leq 32_6\), (iv) \(203_4 \leq 121_6\), (v) \(331_4 \leq 142_6\). (a) Some children detected a pattern and used this rather than think about each comparison. What is the pattern Miss Brill fell into and how could this be corrected? (b) Many children felt overwhelmed by the task and simply guessed. How could Miss Brill have chosen the comparisons that would have fostered reflection, rather than guessing?

3. A student asked, "Can you have a base one?" Is this possible? Why or why not?

4. During the conceptual phase of multidigit addition instruction, Mr. Adams stressed that "no column in a concrete model could have more than nine items." During the connecting phase, the class discussed how each step of the written algorithm corresponded to a step with the concrete model. Deron's answers on the first assignment involving written multidigit addition was entirely wrong. Analyze his work below. Can you identify his systematic error and its likely cause?

5. Some U.S. textbooks introduce two-digit addition and subtraction without renaming even before single-digit addition with sums greater than 10 and their subtraction complements. (a) What do you suppose is the rationale for this? (b) Does this make sense if a meaningful approach (as advocated by the Student Guide) is used?

6. Eric's teacher was puzzled by his inconsistency. He correctly computed the first four sums, missed the next three, and got the last—and arguably the most difficult—sum correct. Analyze Eric's work and devise a hypothesis about his errors.

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7. A trick for multiplying a two-digit number by 11 is: (1) Put the tens-column digit on the left; (2) put the ones-column digit on the right; (3) add the tens- and ones-column digits together and place this sum between the two digits already recorded. Figure A illustrates the trick with 43 x 11. When Step 3 results in a two-digit sum, record the ones digit in the middle (in the tens place) and add the tens digit to the left-hand digit (hundreds place). Figure B illustrates this adjustment with 49 x 11. (a) Why does this trick work? (b) Can you devise a trick for multiplying three-digit numbers by 11? (c) Can you devise a trick for multiplying four-digit numbers?

8. Consider a system that uses the fingers of the right hand to represent one to five. For example, five is represented by putting up all the fingers on the right hand; six, by putting up one finger on the left hand and one on the right; seven, by putting up one finger on the left hand, two on the right, and so forth. Is this a base-five system? Why or why not?

SUGGESTED ACTIVITIES

1. Using references such as Historical Topics for the Mathematics Classroom published by NCTM and The Historical Roots of Elementary Mathematics (Bunt, Jones, & Bedient, 1976), Internet resources such as "MacTutor History of Mathematics Archive" (http://www-groups.dcs.st-and.ac.uk/~history/index.html), and other resources, research the historical development of
(a) Hindu-Arabic numerals, (b) an ancient numeration system, (c) a multidigit computation procedure currently used in this country or abroad, or (d) a computation procedure used in ancient times. After checking with your instructor for guidelines, write a report summarizing what you have found. Share your report with your class by (i) giving an oral presentation, (ii) creating a bulletin board, or (iii) developing a web site complete with text and graphics.

2. Interview about four first-, second-, or third-grade classroom on their grouping and place-value knowledge. (a) One option is to use a standard test such as the Test of Early Mathematics Ability: Second Edition (TEMA-2, Ginsburg & Baroody, 1990). Useful TEMA-2 items would include 25 (Reading Numerals: Teens), 26 (Writing Two-Digit Numerals), 28 (Counting by Tens), 29 (Reading Numerals: Two Digits), 36 (Count By Tens Over 100), 38 (Reading Numerals: Three Digits), 39 (Writing Three-Digit Numerals), 41 (Tens In One Hundred), 46 (Hundreds in One Thousand), 53 (Reading Numerals: Four Digits), 55 (Written Addition Accuracy: Carrying), 56 (Written Addition Procedure: Carrying), 59 (Smallest and Largest Digits, 62 (Written Subtraction Accuracy: Borrowing), 64 (Written Subtraction Procedure: Borrowing). After familiarizing yourself with the rationale for the items and the instructions for administering the items, practice the test items by giving them to a colleague. After interviewing the children, analyze your results and write up a profile on each child—including his or her strengths and weaknesses. Include in your report specific instructional recommendations based on chapter 6 of the Student Guide. (b) Alternatively, devise your own test items. See research reports such as Miura (1987) and Ross (1989) for ideas about items. After interviewing primary-level students, write a report on what you found. Include descriptions and drawings of the test items and graphs of the results.

3. (a) Develop a purposeful, meaningful, and inquiry-based unit on base-ten, place-value concepts and skills for a grade level of your choice (see, e.g., Activity Files 6.1, 6.3, and 6.4 on pages 6-7, 6-15, and 6-18, respectively, in the Student Guide). Specify the content aims (e.g., children will invent a concrete procedure for adding multidigit numbers), any process aims (e.g., working together cooperatively), the worthwhile task(s) that will prompt student explorations, and a general plan of how to proceed (e.g., how a teacher might create cognitive conflict and help students resolve it). Indicate how the unit is consistent with an investigative approach. (b) Implement the unit with an elementary class and assess your pupils’ learning. (c) Evaluate your unit, including how it could be improved. (d) Write a report about how the children responded to the unit, including their discoveries or inventions and what difficulties they experienced.

4. (a) Use 3- x 5-inch cards to make up a deck of cards for War In Other Bases (WOB). The game is played like the card game War except that students would have to determine, for example, if 345, 122, 1011, or 2024 is the winning card. (b) Try the game with others in your group. (c) Develop a unit plan for introducing an elementary-level class to other bases. Include WOB in your plan. (d) Implement the unit and evaluate your effort.

5. (a) Using the bulletin board on page 6-1 of the Student Guide or Investigation 6.1 on pages 6-4 to 6-6, devise a lesson plan for introducing a small group or a class of elementary-level children to an ancient numeration system. (b) Implement your lesson plan and evaluate the effectiveness of it.

6. (a) Introduce a class of elementary-level children who have not had prior multidigit-addition instruction to one of the trading games described in Activity File 6.2 on page 6-8 of the Student Guide. Have the children work in small groups of about four, and encourage them to devise their own concrete or symbolic procedure for keeping a running score of the game. Observe their efforts for at least two sessions. Write a report on how the children responded to the challenge of devising a scoring procedure. Answer the following questions in your report: Were there groups that were able to devise a workable scoring system without help? If so, how efficient was the scoring system? Did the model involve grouping, place value, or both? In terms of the types of models described on page 6-10 of Probe 6.1 in the Student Guide, what type(s) of
concrete models did the children invent? Did any of the groups invent more efficient procedures as time went on? If so, what was the nature of the improvement? If children invented a symbolic procedure, explain how it is less, equally, or more efficient than the traditional right-to-left renaming algorithm taught in U.S. schools.

7. Identify two or more children having difficulty learning a multidigit algorithm. (a) Consider how you could create cognitive conflict to promote reflection and learning (see, e.g., Box 6.3 on page 6-20 of the Student Guide). (b) Implement your plan, videotape the lessons, and evaluate its effectiveness.

8. (a) Create a situation that involves multidigit multiplication or division. Prompt small groups of children who have not had formal instruction on the topic to devise their own procedures. (b) Write a report that describes the children’s efforts.

9. (a) Devise a lesson on multidigit arithmetic based on a children’s book (see, e.g., pages 6-30 and 6-31 of the Student Guide). (b) Present the lesson to a small group or class of elementary-level children, videotape the lesson, and evaluate its effectiveness.

10. Try deciphering the following systematic errors and predicting how the child would respond on the last two items.

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(a) For correct items, ask children how and why they did what they did. (b) For incorrect items, do the same and try to determine how the child arrived at the answer. Note whether a child consistently made a systematic error, (e.g., always subtracted smaller from larger), consistently used a combination of systematic errors (e.g., used a 0 - n = 0 rule when subtracting a number from 0 but otherwise renamed without reducing), or inconsistently used incorrect or partially correct procedures (e.g., used a subtract smaller from larger rule on some not all opportunities to do so). (c) Devise additional test items that would enable you to distinguish between the systematic errors that you do find—that would result in different answers for each systematic error.

HOMEWORK OR ASSESSMENT

QUESTIONS TO CHECK UNDERSTANDING

1. Circle the letter of any of the following statements that, according to the Student Guide, is true. Change the underlined portion of any incorrect statement to make it true.

a. The most common error made by primary-level children when writing multidigit numerals is to write the digits in reverse order (e.g., writing "forty-two" as 24).

b. Three major barriers to understanding base-ten place-value concepts are (a) instruction has traditionally focused on mastering multidigit reading and writing skills, (b) English counting terms disguise the grouping-by-ten meaning of multidigit numerals, and instruction has traditionally been too abstract.

c. Building on informal knowledge is not useful in teaching place-value numeration skills/concepts.

d. The Egyptian hieroglyphics are an excellent teaching aid, because they are a straightforward place-value system.
e. To set up a chip-trading game for a group of four children in BASE FOUR, with a winning score of 75, a teacher would need 3 different-colored chips to represent ones, tens and forty.

f. To set up a chip-trading game for a group of four children in BASE FIVE, with a winning score of 59, a teacher would need 3 different-colored chips to represent ones, tens and fifty.

g. The equation $23 + 42 = 65$ is possible in base 6, base 8, base 10, or base 12.

h. Working with a concrete renaming (carrying) procedure for addition should be done well after instruction on reading and writing multidigit numerals and base-ten place-value concepts.

i. Pedagogically and psychologically, it makes sense not to expose children to a variety of written multidigit subtraction procedures.

j. A groups-of meaning can help children make sense of multidigit multiplication.

k. The formal distributive long-division algorithm follows from a measure-out interpretation of division.

l. The formal subtractive algorithm symbolically parallels and represents a divvy-up interpretation of division.

3. What numeral would represent the collection in (a) base ten, (b) base two, (c) base seven, and (d) base sixteen?

4. Change each of the following base-ten numerals to base-seven numerals: (a) 9, (b) 285, and (c) 1000.

5. Write each of the following as a base-ten numeral: (a) $157_g$, (b) $B6_{12}$, and (c) $504_b$.

6. A large cube in a base-three block set would have what base-three value?

7. Consider the transition points in our base-ten system: nine (9), ninety-nine (99), nine hundred ninety-nine (999), nine thousand nine hundred ninety-nine (9,999), and so forth. Now consider what the transition points would be in base 3. Use this knowledge to deduce the answer to the following subtrac-
tion problem written in base 3. (Write your answer in base 3.)

\[
\begin{array}{c}
1,000,000,000,000,000,000,000,000,000,000,000 \\
- 1
\end{array}
\]

8. (a) For each of the following pairs of numbers circle the larger. (b) Put in the correct relational symbol: =, >, or <.

a. \(110_7\) ___ \(210_5\)  c. \(1D_{100}\) ___ \(414_5\)
b. \(1B_{13}\) ___ \(222_3\)  d. \(43_5\) ___ \(10111_2\)

9. All the questions below refer to base 8.

a. The base-8 system has how many basic building blocks (single-digit symbols)?

b. How would a set of 74 items be represented in base 8?

c. Would the numeral 18 ever appear in base 8? If so, what (base-ten) quantity does it represent?

d. Would the numeral 100 ever appear in base 8? If so, what (base-ten) quantity does it represent?

10. All the questions below refer to base 13. Write your answer in base 10 unless otherwise instructed.

a. How many basic building blocks (single-digit symbols) does the base 13 system have?

b. How is a set of thirteen represented in base 13? (Write answer in base 13.)

c. How many items are represented by the base 13 numeral 1C?

d. How would a set of 170 items be represented in base 13? (Write answer in base 13.)

e. Would the numeral 13 ever appear in base 13? If so what quantity does it represent?

f. Would the numeral 1D ever appear in base 13? If so what quantity does it represent?

---

g. In the solution to the right, what does the 1 above and to the left of the 2 represent?

\[
\begin{array}{ccc}
-835_{13} & \times & 3 \\
\hline
-942_{13} & + & -835_{13} \\
\hline
-2420_{13}
\end{array}
\]

---

h. What is the answer to the problem above written in base 13?

11. My Age?—How Base

a. Miss Brill was going to be 24 on her next birthday. She thought that it might be fun if her class figured out how old she would be in different bases. She challenged her class to find her new age in base four, five, and six. How old would Miss Brill be on her next birthday in these bases?

b. Meg was feeling very depressed about turning the Big 4-0. Hazel tried to cheer Meg up by pointing out her age in base 12. How old is a 40-year-old in base 12?

c. If Hazel had been a really good friend, she would have picked a base system that made Meg 18-years old. In what base system would a 40 year-old be 18?

d. Professor Magnus was complaining about how old he felt turning 60 years old. Larisa, his graduate student, pointed out: "But think how you would feel if we used base two." Is there any point in Larisa continuing her graduate work with Professor Magnus? How old would Professor Magnus be in base two anyway?

12. a. In base ten, the written equation \(5 + 5 = 12\) is obviously incorrect. But can \(5 + 5\) (ooooo + ooooo) ever equal 12? That is, is there a base system in which the written equation \(5 + 5 = 12\) is correct? Briefly justify your answer.

b. Is there a base system in which the equation \(5 + 5 = 7\) is correct? Briefly explain.

c. Is there a base system in which each of the following written equations is correct? Briefly justify your answers.

(i) \(5 + 5 = 22\)  (iv) \(5 + 5 = 13\)
(ii) \(5 + 5 = 20\)  (v) \(5 + 5 = 11\)
(iii) \(5 + 5 = 14\)  (vi) \(5 + 5 = A\)
d. Why might children (and many adults) be baffled that any one would even pose a question like Question 12a? That is, why might they think the answer obviously is no?

e. Would the answer to Question 12a change if it had been posed: "Can five plus five ever equal twelve?" Consider what twelve implies that the multidigit numeral 12 does not.

13. Your student teacher reads the equation $24 + 24 = 104$ as "two plus two is ten" and the equation $234 + 14 = 304$ as "twenty-three plus one equals thirty." The students were very confused. Why? What should you point out to your student teacher so that she does not repeat her error?

14. Indicate in what base (or bases) each of the following sets of problems are written.

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</table>

15. What technique below best illustrates how a teacher using the investigative approach would teach multidigit arithmetic algorithms?

a. Illustrate with a manipulative how each step of a formal algorithm can be modeled and, thus, makes sense.

b. Explain each step of an algorithm in terms of a familiar and meaningful analogy.

c. Relate each step of an algorithm to a meaningful analogy that is also modeled concretely with manipulatives.

d. Encourage children to reinvent written algorithms by representing symbolically each step of self-devised concrete models.

e. Permit children to engage in free play and hope they spontaneously rediscover the written algorithms.

f. Focus on memorizing by rote the standard algorithm (step-by-step procedure) for an operation.

16. Which of the following is the most concrete model for the renaming process (note with a C)? Which is the most abstract—that is, most closely resembles the written (symbolic) carrying procedure (note with an A)?

   a. High Card (described in Activity File 6.2 on page 6-8 of the Student Guide) using colored chips and a 10-for-1 trading process to keep score (when 10 whites are traded in for 1 red, which equals a ten, and 10 reds are traded in for 1 blue, which equals a hundred).

   b. High Card using base-ten blocks (10 small cubes are traded in for 1 long, a ten, and 10 longs are traded in for 1 flat, a hundred).

   c. High Card using 10-for-1 trading and same-colored blocks, which by their position denote ones, tens, hundreds.

17. Demonstrate or illustrate how a child might use base-ten blocks to concretely model a groups-of interpretation of (a) 4 x 58, (b) 12 x 58, and (c) 23 x 58.

18. a. Demonstrate or illustrate how base-ten blocks could be used to create an area model of the multiplication sentence $24 \times 15 = ?$

   b. Compare this concrete representation to the partial products of the expanded partial-products multiplication algorithm. What do you notice?

   c. Illustrate on graph paper an area model of $14 \times 15$. Show how this area can be broken up so that it takes advantage of children's knowledge about multiples of 10.
19. a. Demonstrate or illustrate how base-ten blocks could be used to create an area model to solve the division sentence $286 \div 22 = ?$

b. Describe and illustrate how base-ten blocks can be used to model and to explicitly explain each step in the following algorithmic solution.

\[
\begin{array}{c|c}
13 & 286 \\
22 & -22 \\
\hline
66 & \\
-66 & 0 \\
\end{array}
\]

20. Mr. Nowling gave his class a problem that involved, in part, dividing 1829 by 27. A common error he found is illustrated below.

\[
\begin{array}{c|c}
67.20 & 78.34 \\
27\overline{1829} & 35\overline{2764} \\
162 & 245 \\
209 & 314 \\
183 & 280 \\
20 & 34 \\
\end{array}
\]

(a) Describe the systematic error. (b) To create cognitive conflict so that students might reconsider their incorrect procedure, Mr. Nowling had them check their work. Briefly describe two ways this could be done.

21. Hattie’s work below illustrates another common error.

\[
\begin{array}{c|c}
517 & 618 \\
27\overline{1829} & 35\overline{2764} \\
135 & 210 \\
47 & 66 \\
27 & 35 \\
209 & 314 \\
\end{array}
\]

(a) Describe the systematic error. (b) How could a teacher help Hattie to see that her answers do not make sense?

**WRITING OR JOURNAL ASSIGNMENTS**

1. Ms. Fuzz, one of your colleagues, comes back from a workshop quite excited about a “new” method for introducing multidigit concepts, namely the abacus. Evaluate this recommendation in terms of (a) providing the most concrete model possible for introducing a grouping concept and (b) for introducing place-value concepts. (c) What conclusions about using an abacus for teaching numeration concepts can you draw?

2. Henry just can’t seem to understand the multidigit renaming procedures. (a) Ms. Peach decided she needed to check whether Henry had constructed a deep understanding of place-value. She reasoned that Henry must develop an understanding of grouping and place value before he is introduced to the written multidigit renaming procedures. According to the Student Guide, is Ms. Peach on solid ground? Briefly justify. (b) Ms. Peach found that Henry understood diddley about base-ten, place-value concepts. She decided Henry needed to start from square one: work with a really concrete model. She had read about a chip-trading game where a child collects 10 white chips and trades them in for a red chip. She decided to have Henry play chip-trading games. Evaluate Ms. Peach’s instructional decision.

3. (a) Explain how base-ten blocks could be used to model each step in the standard renaming algorithm for addition, using $138 + 65$ as an example. (b) Explain how Egyptian hieroglyphics could be used to model each step in the standard renaming algorithm for subtraction using $402 - 153$ as an example.

4. A student once asked, “With [the standard multidigit] addition, subtraction, and multiplication [algorithms], you start with the ones digit. With [the standard multidigit] division algorithm, why do you start with the left-hand most digit [of the dividend]?” Why does the long-division procedure begin with this digit rather than the one digit?

5. Box 6.2 (on page 6-20 of the Student Guide) illustrated how student analyses of correct and
incorrect procedures can provide a basis for indirectly introducing the standard multidigit addition algorithm. Describe how this investigative approach could be used to introduce the standard two-digit multiplication algorithm.

6. While solving a problem, Kutar performed the computation shown to the right. According to the teaching tips outlined in chapter 6 of the Student Guide, how might a teacher help Kutar correct his faulty procedure?

7. The distributive long-division algorithm repeats the following sequence of steps: divide, multiply, subtract, and bring down. How could a teacher help a student understand the multiply and subtract steps? That is, specify why these steps are done and how they contribute to determining the quotient.†

8. While student teaching, your cooperating teacher asks you to help Arlette, who is having trouble with the long-division algorithm. (a) What meaning of division can you draw on to help Arlette understand the algorithm. (b) Arlette comments, “The thing I don’t understand about this long-division thing is why you have to multiply and subtract.” How can you use a meaningful analogy and base-ten blocks to help Arlette understand why these operations come into play. (c) Arlette also comments, “When the teacher did this problem on the board she got R3. What does R3 mean?” What can you tell Arlette?

9. After discussing your investigative approach to multidigit division, one of your colleagues asks, “How can you expect children to invent their own procedures for something so complex as multidigit division? Even if they could, why waste precious class time on discussing and sharing informal methods? Isn’t it important to learn the standard algorithm, which after all is efficient? How will inventing their own procedures help them learn the standard algorithm?” How would you address each of these questions?

† Based on task described by Martin A. Simon (1993).

PROBLEMS

- Three-Digit Numbers (4-8)

  How many three-digit numbers can be made from 1, 2, 3, and 3?

- Frequent Fours (7-8)

  a. How many whole numbers between 0 and 100 begin and/or end with a 4?

  b. How many whole numbers between 300 and 500 begin and/or end with a 4?

- Place-Value Riddle (4-8)

  If a 3 is written on the right side of this two-digit number, the value of the new three-digit number is increased by 300 over the original number. What is the original two-digit number?

- The Last Page (7-8)

  Numbering the pages of a book required 2997 digits. What is the page number of the last page?

- Thinking Strategies in Other Bases? (6-8)

  The Times-Nine Thinking Strategy specifies the digits of the product of nine and a number must sum to nine (see pages 5-32 and 5-33 in the Student Guide for a more complete description and an example). Is there a comparable thinking strategy in other base systems? Consider, for example, bases 4 and 9.

- Rationale for Times-Nine Thinking Strategy (6-8)

  The Times-Nine Thinking Strategy specifies the digits of the product of nine and a number must sum to nine (see pages 5-32 and 5-33 in the Student Guide for a more complete description and an example). Why does Times-Nine Thinking Strategy work? Consider how you could use base-ten blocks to model the multiplication of 7 x 9, for example, and to show why the sum of the digits of this product is nine.

- The Base-Conversion Mystery (6-8)

  Mr. Yant asked his eighth graders to give him a base-two number. Josephine offered 11011101. Mr. Yant wrote down the digits of the base-two
number and rather quickly announced that it was equivalent to 673₈. After laboriously converting the base-two number into a base-ten number and doing the same for the base-eight number, the students found that both numbers indeed equaled 443 (base 10). If the students were amazed by how quickly Mr. Yant could convert base 2 numbers to base 8 numbers, they were even more amazed by how quickly he could convert base 8 numbers to base 2. For example, given 735₈, Mr. Yant—without even writing down the digits—announced, "111011101 base two." He then challenged his class to find an explanation for his amazing proficiency. What is the basis for Mr. Yant’s proficiency? Hint: Consider the place-value system of base-two and base-eight numbers. Could Mr. Yant’s feats be duplicated with base four and base two? Why or why not? Could his feats be duplicated with base five and base two? Why or why not?

Play Ball, Basketball That Is (◆ 6-8)

Twenty-five boys are in the gym and want to play basketball. They have the use of the gym for 40 minutes, but only 10 can play at a time. If each is to play the same number of minutes, how many minutes should each play?

The Meter Is Ticking (◆ 6-8)

A meter has only 6 markers. The pointer moves clockwise adding one unit to the counter each time it reaches a marker. Both the pointer and the digital counter start at zero. (Note that the dials on the digital counter run from 0 to 9.) Which number will the pointer be at when the digital counter reaches 400? Explain why.

Conflict Over the Lap Counter (◆ 6-8)

Ms. Socrates challenged her class with the following word problem:

Lap Counter. Mr. Rao swam 25 laps for exercise. Unfortunately, he kept forgetting to count laps or otherwise lost track of his count. He decided to use toy diving sticks to count off laps. If Mr. Rao marked each lap with a stick, grouped the sticks by five and used a place-value system, what is the fewest number of sticks he would need?

Rowland concluded 25 were needed; Yi-Tze, 10; Khalada, 9; Abuka, 8, and Murithi, 5. Who, if anyone, was correct? Justify your answer.

Chip-Trading Problems (◆ 6-8)

In answering the following question, assume that all the children playing are naive—that is, they will collect the number of white chips shown on a die roll and only then trade in. Assume that the children will trade in when they have the opportunity to do so. Assume a 0 to 5 dot die is used.

Situation 1. In chip trading that involves regrouping by five, what is the fewest number of whites (ones), reds (fives), and blues (twenty-fives) a teacher should provide for a group of four children? Assume that 100* points is the stopping point (i.e., the winner is the first player to get 100 points). Hint: Consider the maximum number of white chips each player could have at the beginning of a turn. Consider the maximum number of white chips that a player could collect on a turn. Do the same for red and blue chips.

Situation 2. How many of each color would the teacher need to provide in Situation 1 if each group had five children instead of four?

Situation 3. If Situation 2 involved a stopping point of 150* instead of 100, how many colors would the teacher need for a group of five children? What will be the value of each color?

Situation 4. For chip trading that involves regrouping by two and a stopping point of 25*, how many colors would the teacher need for a group of five children? What will be the value of each color?

Situation 5. For chip trading that involves regrouping by two and a stopping point of 75*, how many colors would the teacher need for a group of five children? What will be the values of each color?

* A question that sometimes comes up is: Are the stopping points listed in Base 10 or is the base implied in the problem? Examine the stopping points. Could the listed values be in Base 2 or Base 5? Why or why not?
ANSWER KEY for Student Guide

Key for Investigation 6.1 (pages 6-4 to 6-6)

Part I

1. (a) Yes; the symbol \( \overline{1} \), for example, could represent 1 or 60 depending on its position.
   (b) For instance, \( \overline{1} \overline{1} \) (both wedges in the right column) represents 2, while \( \overline{1} \ \overline{1} \) (one wedge in each of the left and right columns, which are separated by a space) represents 60 + 2 or 62.

2. (a) \( 195 = \overline{1}\overline{1}\overline{1} \overline{1} \overline{1} \overline{1} \) \( \overline{1} \) \( (3 \times 60) + 10 + (5 \times 1) \)  
   (b) \( 267 = (4 \times 60) + (2 \times 10) + (7 \times 1) \)  
   (c) \( 371 = (6 \times 60) + (1 \times 10) + (1 \times 1) \)

3. \( 600 = \overline{1} \overline{1} \) in the left-hand column \( (10 \times 60) \)

Part II

2. (a) The common error made by Alexi is probably due to viewing multidigit numbers in terms of a counting-based or unit conception of numbers—e.g., viewing 12 as 12 units rather than as a group of 10 and 2 ones.  
   (b) After writing 12 staffs for 12, a teacher could prompt, "Is there an easier way to show 12 in Egyptian hieroglyphics?"  
   This avoids denigrating children's answer and gives them an opportunity to correct themselves.  
   For some children, it may be necessary to remind them that \( \cap \) (a heel) equals ten.

3. (a) The task (implicitly) involved the idea that position defines value and that a multidigit numeral such as 12 represents 1 group of ten and 2 ones.  
   Translating both 12 and 21 into Egyptian hieroglyphics \( (\overline{1}\overline{1} \text{ and } \overline{1}\overline{1} \overline{1}\overline{1}\overline{1} \overline{1} \overline{1}) \), respectively) can concretely highlight a very important point about our numeration system—position (place) defines value.  
   A comparison of the four representations makes it clear that the 2 in 12 represents 2 ones and the 2 in 21 stands for something different—2 tens.  
   (b) Such a task might help prompt first graders to count-on (rather than count-all).  
   It may also help them see that a number added to 10 is the number + teen (e.g. \( 10 + 7 = \text{seven} + \text{teen} \)), a useful rule that would eliminate the need to count on to determine \( 10 + n \) sums.

4. (a) The ancient Egyptians didn’t need to use zeros because their number system didn’t involve place value.  
   For example, 302 translates into \( \overline{3} \overline{2} \).  
   No (zero) tens is represented in Egyptian hieroglyphics by not writing any heels(\( \cap \)).  
   (b) The question addresses the role of zero as a place holder.  
   After some reflection 8-year-old Alison answered the question by saying, "Well we use a zero because this [pointing to a 13] is thirteen.  
   If we didn't use a zero, how would you write one hundred three?  
   One three?  
   [The ancient Egyptians] didn't need a zero.  
   When they wrote thirteen \( \overline{1}\overline{1}\overline{1} \) and one hundred three \( \overline{1}\overline{1}\overline{1} \), they used different signs.  
   They [the symbols for 13 and 103] looked different."

5. The task was posed to illustrate the elegance of our decimal system.  
   Whereas it took the ancient Egyptians 36 symbols to represent 9,999—a tiresome task as Julius pointed out—it takes us only 4 symbols.  
   Our decimal numeration system is powerful because it can represent large numbers in a relatively economical way.

6. (a) 36.  
   (b) Third graders should recognize that multiplication is applicable: four digits x 9 symbols per digit = 36.  
   Younger children might recognize the applicability of repeated addition.

Part III

1. (a) The Roman numerals were an incomplete place-value system.  
   For example, a I before a V means "one less" and a I after a V means "one more than."

Part IV

The Mayan number system is a base-twenty, place-value system.  
The eye-shaped symbol represents zero and serves as a place holder.  
The first row of symbols indicates the number of ones; the second row, the number of twenties; and the third row, 20 x 20 or four hundreds.

3. Like Mayan (or Babylonian) hieroglyphics, a symbol in our Hindu-Arabic system can be multiplied by a factor to represent a larger number.  
   For instance, 60 in Mayan is literally 3 x 20 (in Babylonian, 1 x 60).

Part V

To decode 7,352,468,329, you first need to determine the value of the left-most digit 7:  Start
with the right-most comma and mentally note that it represents thousands; proceeding leftward, mentally note that the next comma represents millions and that the one after that represents billions (see Figure 6.1 at the bottom of this page). Because there is only one digit before this last comma, you can begin reading the numeral as seven billion (literally, seven one-billions). By exploiting the repeating structure of our decimal system, you can then recognize that the 352 represents three hundred fifty-two millions and that the 468 represents four hundred sixty-eight thousands. Next, you need to read the third digit to the right and couple it with its place designation (three hundred) and finish off by reading the two digits in the tens and ones place as twenty-nine.

Key for Probe 6.1 (pages 6-10 and 6-11)

1. i = C (nonproportional and different-looking markers: a $10 bill, for instance, has different markings than does a $1 bill); ii = D (nonproportional and the position of a disk defines its value); iii = cross between C and D (nonproportional and both color and position define value); iv = D; v = A (proportional model and child creates the group of ten).

2. (a) Model B. (b) Models C, D, i, ii, iii, and iv.


Key for Investigation 6.2 (pages 6-12 and 6-13)

Part I

Base 12
a. 12
b. 2A
c. (11 x 12) + 2 = 134
d. 10A
e. Yes; 14
f. Yes; 17

Base 5
a.
b.
c.
d.
e.
f.

Part III: Swine Identification

2. (a) Base three. (b) $3^4 = 81$. (c) Pig 2 would be represented by two notches in Zone 1 of the pig’s left ear. Litter 15 would be represented by two notches in Zone 2 and one notch in Zone 3 of the pig’s right ear. (d) Pig 7 would have one notch in Zone 1 and two in Zone 2 of the pig’s left ear; Litter 24 would be represented by two notches each in Zones 2 and 3 of the right ear.

Key for Probe 6.2 (page 6-16)

Conceptual Phase

1. (a) The problem is What can you do when you must take away more ones than are available? (b) Solving the problem themselves creates an opportunity to reinvent the renaming (borrowing) procedure. (c) Using other bases to prompt the discovery of the renaming procedures avoids the interference children’s unit concept of number may cause if grouping by 10 were used. Moreover, it can help establish a general understanding of renaming, which can provide a basis for understanding renaming with 10.)

2. Model D, the most abstract model and the only one that involves place value, most directly models the renaming algorithm. Model D not only involves abstract 10-for-1 trading, but the item put in the tens column in exchange for 10 ones is identical in appearance to the items representing ones. (Likewise, the 1 that represents a "carry" is identical in appearance to 1s appearing in the ones place.)

3. Each player could start with, say, 200 points, which would be indicated by two blocks in the hundreds column and no blocks in the tens or ones column. Points scored would be taken from a player’s total, and the first player to reach 0 would win. If a player scored eight points or her first turn, for instance, she would have to remove eight blocks from the ones

<table>
<thead>
<tr>
<th>Figure 6.1: Decoding a Multidigit Number</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>billion</strong></td>
</tr>
<tr>
<td>hundreds</td>
</tr>
<tr>
<td>7,</td>
</tr>
</tbody>
</table>
column. This poses a problem because these are no ones. With any luck, students will solve the problem by trading a block in the hundreds place for 10 blocks put in the tens place and then trading a block in the tens place for 10 blocks put in the ones place.

**Key for Investigation 6.3 (page 6-17)**

1. Illustrated below is the following procedure for adding 389 + 155 using Egyptian hieroglyphics: (1) represent each addend; (2) begin by adding the staffs (ones); (3) if adding results in a group of 10, trade the 10 staffs (ones) in for 1 heel (a 10); (4) repeat this process for the heels, and coils.

**Figure A**

![Figure A](image)

**Figure B**

![Figure B](image)

**Key for Probe 6.3 (pages 6-25 and 6-26)**

**Part I**

1. Divvy-up. The amount and number of shares are known, and the size of the shares is not.

**Part II**

1. Measure-out. The amount and the size of the shares are known, and the number of shares is not.

2. Repeated subtraction.