TEACHING TIPS

AIMS AND SUGGESTIONS

Units 5•1 and 5•2: Introducing the Four Arithmetic Operations

One key aim of Units 5•1 and 5•2 is helping readers to explicitly recognize (a) the various meanings an operation can have and (b) why this is important for children to learn this variety of meanings. The vast majority of adult students are not consciously aware that an arithmetic operation can represent a variety of real-world situations or meanings. Moreover, they typically do not realize that equating an operation with a single meaning can limit children. For example, because some children do not recognize “difference” situations (compare or equalize problems) as subtraction, they do not use their known subtraction combinations to determine their answers to such situations. As also discussed in the Student Guide, a limited view of multiplication and division can create difficulties in understanding fraction or decimal multiplication or division. Part I of Probe 5.1: Addition and Subtraction Word Problems (page 5-6 of the Student Guide), Probe 5.2: Meanings of Multiplication (page 5-15), and Investigation 5.2: Explicitly Relating Division to Multiplication Meanings (pages 5-18 an 5-19) were designed to help students consciously reflect on the various meanings of the four operations.

A second key aim is to underscore the fact that children spontaneously invent their own informal strategies to solve arithmetic problems before they receive formal arithmetic training. A key pedagogical implication of this fact is that arithmetic instruction should start with problems and that teachers can encourage children to use their informal knowledge to devise their own solutions. Part II of Probe 5.1 (pages 5-6 and 5-7 of the Student Guide), Probe 5.3: Modeling the Various Meanings of Multiplication (page 5-16), and the Questions for Reflection of Investigation 5.2 (page 5-19) were designed to help readers better appreciate the power of children’s informal knowledge and the elegance of their informal arithmetic strategies. Part III of Probe 5.1 (page 5-7) can help underscore the role of word problems in a mathematics curriculum and the importance of introducing children to a variety of meanings.

A third key aim is to help readers recognize the importance of connecting formal arithmetic to children’s informal knowledge and connecting concepts to other concepts. Because many students were taught mathematics in a traditional manner in which instruction started immediately (or almost so) with symbolic arithmetic and concepts were typically taught in isolation, the ideas above may be novel to them and, thus, need emphasis. The following reader inquiries can help with this process: Investigation 5.1: Using Concept Maps to Explore Arithmetic Concepts (page 5-10 of the Student Guide), Probe 5.A: Analysis of an Introduction to Multiplication (page 122 of this guide), Investigation 5.2: Explicitly Relating Division to Multiplication Meanings (pages 5-18 and 5-19 of the Student Guide), Probe 5.4: Helping to Make Division Meaningful (page 5-23 of the Student Guide), and Investigation 5.A: Using Concept Maps and Inductive Reasoning to Explore Division (pages 123 and 124 of this guide).

Unit 5•3: Mastery of Basic Combinations

The key aim of the Unit 5•3 is to help readers understand that mastering the basic number combinations in a meaningful manner is ultimately a far more effective process than mastery of basic facts by rote. Part I of Probe 5.5: Mastering the Basic Number Combinations (page 5-25 of the Student Guide) can help underscore the unconventional (but research-supported) belief that combination mastery, even among adults, can involve a variety of strategies: fact recall, automatic or semi-automatic rule-based or reasoning processes, and—perhaps even—rapid counting (Baroody, 1985; Browne, 1906; LeFevre, Bisanz, Daley, Buf-
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fone, Greenham, & Sadesky, 1996a; LeFevre, Sadesky, & Bisanz, 1996b). **Part II** of this probe is intended to prompt reflection on how a teacher can build on what children already know to help them reason out unknown combinations. **Part III** underscores the important role looking for patterns and relationships can play in mastering the basic number combinations.

**SAMPLE LESSON PLANS**

**Project-Based Approach**

Using the SUGGESTED ACTIVITIES on pages 121, 125, and 126 of this guide as a menu, have students or small groups of students choose a project. Note that Activities 1 and 2 can help deepen their understanding of arithmetic meanings. Activities 1 and 3 can help adult students reflect on the connection between school arithmetic and arithmetic in the real world. Activities 4 an 5 can help them consider how games can play an important role in arithmetic instruction and practice. Activity 4 is also technology related. Activities 6 and 7 require students to critically analyze and evaluate textbooks. Activities 8 and 9 would involve students in assessing, planning instruction for, and teaching a child—actually applying chapter 5 material.

**Single-Activity Approach**

**Lessons 1 and 2.** To prompt students to familiarize themselves with the various meanings of the operations and the connections among them, have them work in small groups to construct a concept map involving the four operations. One way to proceed is to begin with **Investigation 5.1: Using Concept Maps to Explore Arithmetic Concepts** (page 5-10 of the Student Guide). After discussing the groups' concept maps of addition and subtraction as a class, have students add multiplication meanings to their maps by completing **Part II of Probe 5.2: Meanings of Multiplication** (page 5-15 of the Student Guide). After discussing this addition as a class, have students add division meanings to their maps by completing **Part I of Investigation 5.A** (page 123 of this guide).

Constructing a concept map should prompt students to reflect on, discuss, read about, and otherwise learn more about the various meanings each operation has and how the operations are interrelated. Encourage students to use a pencil to work on the first drafts of their maps. After a group or class discussion, have them devise a final version in ink or magic marker. Prompt students to write descriptive and precise linking phrases instead of simply drawing lines that link concepts or using one-word, nondescriptive terms such as includes. Many students may find concept mapping to be hard work, because it requires them to explicitly and precisely define concepts and the relationships among them. Although concept mapping is arduous, it can help adult students construct a clearer understanding of the concepts, putting them in a better position to help children build a clear understanding of the concepts. An instructor may use the class discussion of maps as a basis for creating cognitive conflict and as a reminder that there may be various ways of constructing an accurate map.

**Lesson 3.** To highlight the phases of number-combination learning, engage the class in the following alphabet-arithmetic activities:

1. To explore how basic number combinations are learned, researchers have used alphabet codes (e.g., A = 1, B = 2, C = 3, D = 4, and so forth). Adult subjects are asked to learn the basic number combinations in the alphabet code (e.g., A + A = B, A + B = C, and B + C = E). (a) *Without* translating the letters into numbers, use the alphabet code illustrated above to determine the alphabet sum of D + E. (b) How did you figure out the sum? Compare your strategy with that used by others.

2. a. If A = 1, B = 2, C = 3, and so forth, what is the (alphabet) answer to the following combinations:

   (i) D + A  (ii) H + A  (iii) X + A  
   (iv) A + M  (v) A + R  (vi) A + Y  
   (vii) J - A  (viii) Q - A  (ix) F - A  
   (x) T - A  (xi) Y - A  (xii) Z - A

   b. Were these relatively easy or difficult combinations to figure out? Why? Specifically, how could someone determine the answers to combinations i to vi and vii to xii quickly—without having to extensively practiced these combinations?

3. If A = 1, B = 2, C = 3, D = 4, and so forth, what letter would complete the following alpha-
betic-arithmetic equation: D + D = ? Take a moment and memorize this alphabet sum. Then complete the following alphabet-arithmetic equations: (a) D + E = ; (b) E + D = ; (c) H - D = ; (d) B x D = ; and (e) H ÷ D = ? Analyze and note how you solved each equation above. Discuss your observations with your group or class. What mathematical relationship could you exploit if you needed to memorize the six alphabet equations above?

4. (a) Most people went through what phases in their ability to generate the sum to combinations such as B + C and C + D? That is, initially most people would do what? Before memorizing B + C = E and C + D = G, many might devise what shortcut(s)? (b) The activities above have what pedagogical implications for fostering mastery of the basic number combinations by children?

Multiple-Activities Approach

Lesson 1. To foster a relatively broad understanding of addition and subtraction and their meaningful instruction, an instructor might use the following sequence of activities:

1. Use Parts I, II, and III of Probe 5.1: Addition and Subtraction Word Problems (pages 5-6 and 5-7 of the Student Guide) to prompt reflection and discussion about types of addition and subtraction meaning, children's self-invented and concrete strategies for modeling these various meanings, and the role problems should play in arithmetic instruction. In Part I, some students have difficulty distinguishing between change add-to and part-part-whole problems (Problems 2 and 3, respectively). The former involves a physical action: An initial amount is increased (changed) by adding more of something. With the latter, no physical action is implied: Both parts of a whole are present initially and their total equals the whole. Students may also have difficulty distinguishing between the two difference meanings of subtraction: compare and equalize meanings (Problems 4 and 5, respectively). The former entails finding the difference and implies no physical difference. The latter involves physically eliminating the difference.

Part II can be valuable because it can help students understand how young children can solve problems without any prior formal instruction: They model the meaning of a problem. This understanding may help them realize that even young children are not helpless. Part III can help underscore that formal knowledge develops later than concrete informal competence or even mental informal competence.

2. Doing the concept map suggested in Investigation 5.1: Using Concept Maps to Explore Arithmetic Concepts (page 5-10 of the Student Guide) as a class can help students explicitly reflect on the meanings of addition and subtraction, their properties, and the connections between the two operations (see Questions for Reflection, Question 2 in particular).

3. Playing Names-for-a-Number Game (Part I of Probe 13.1 on page 13-9 of the Student Guide) can serve to help students to see that (a) a number has an infinite number of representations including those involving arithmetic (e.g., seven can be represented by 6 + 1, 10 - 3, 14 ÷ 2), (b) games can be a useful way of exploring patterns and applying arithmetic principles such as commutativity or practicing basic number combinations, (c) games can be useful in raising questions (e.g., Is a scoring procedure fair?) and exploring issues or content (e.g., averages as a way of taking into account unequal size teams). Moreover, playing this enjoyable game can help foster a positive disposition toward learning and teaching mathematics.

4. After discussing children's informal operator view of equals (= means, e.g., adds up to or produces), challenge students to consider how a Math Balance (see Activity File 5.9: Concrete Equations on page 5-34 of the Student Guide) could be used to foster a relational meaning (= means the same number as). Have them consider how it could help children discover or construct concepts such as commutativity and solve missing-addend problems concretely.

Lesson 2. To foster a relatively broad understanding of multiplication and division and their meaningful instruction, an instructor might use the following sequence of activities:

1. Use Part I of Probe 5.2: Meanings of Multiplication (page 5-15 of the Student Guide) to help students recognize the various meanings of multiplication. Note that some problems might be interpreted in more than one way. For example,
**Lesson 3.** To foster a relatively broad understanding of the teaching and learning of basic combinations, an instructor might use the following sequence of activities:

1. Use Part I of **Probe 5.5: Mastering the Basic Number Combinations** (page 5-25) as a rationale for and an overview of the three-phase approach recommended in the *Student Guide*: counting phase, reasoning phase, and mastery phase.

2. To illustrate the counting phase, introduce the finger-counting methods for determining products shown in **Box 5.5** on page 5-30 of the *Student Guide*. Actually having students try these methods is far more effective and interesting than simply describing or demonstrating them.

3. To illustrate the reasoning phase, have students complete Parts II and III of **Probe 5.5**.

4. To illustrate that the practice needed for mastery can be purposeful and entertaining, play **Crypto** described in **Activity File 5.7** (page 5-29 of the *Student Guide*).

**SAMPLE HOMEWORK ASSIGNMENTS**

**Lesson 1**

Read: Unit 5•1 in chapter 5 of the *Student Guide*.

Study Group:

- **Questions to Check Understanding**: 1, 2, 3, 4, and 5 (Cases 1 to 6) (pages 126 to 128).

- **Problem**: A Number Riddle (page 132).

- **Bonus Problem**: Brother, Can You Spare a Tire? (page 132).

Individual Journals: Writing or Journal Assignment 1 (page 131).

**Lesson 2**

Read: Unit 5•2 of chapter 5.

Study Group:

- **Questions to Check Understanding**: 5 (Cases 7 & 8), 9, 10, 11, 12, and 14 (pages 128 and 129).
• **Problem**: A Nonroutine Division Problem (page 132).

• **Bonus Problem**: Calculated Remainder (page 132).

**Lesson 3**

Read: Unit 5•3 of chapter 5.

Study Group:

• **Questions to Check Understanding**: 15a to 15f, 17, 18, and 19 (pages 129 and 130).

• **Problem**: Rationale for Times-Nine Thinking Strategy (page 162).

• **Bonus Problem**: Thinking Strategies in Other Bases (page 162).

**FOR FURTHER EXPLORATION**

**ADDITIONAL READER INQUIRIES**

Probes 5.A (page 122)

**Analysis of an Introduction to Multiplication**

entails critically analyzing the developmental appropriateness of sample textbook lessons and can serve to underscore that arithmetic instruction should take into account and connect with children's existing knowledge.

Investigation 5.A (pages 123 and 124)

**Using Concept Maps and Inductive Reasoning to Explore Division**

can underscore the importance of connecting formal instruction to conceptual knowledge.

**QUESTIONS TO CONSIDER**

1. One of Miss Brill's students suggested that the difference between the commutative principle and the associative principle of addition (or multiplication) is that the first involved just two terms and latter involved at least three terms. Evaluate this conjecture.

2. For each of the following that reflects the commutative principle of addition, write Ca; for each that reflects the associative principle of addition, write Aa; for each that reflects the commutative principle of multiplication, write Cm; and for each that reflects the associative principle of multiplication, write Am.

   a. \( 9 (4 + 2) = 9 (2 + 4) \)
   b. \( 8 (5 + 3) = (5 + 3) 8 \)
   c. \( 3 (7 + 5 + 5) = 3 (7 + [5 + 5]) \)

3. Melissa May informally rearranged the problem \( 3 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 5 \) as \((3 \cdot 3) (5 \cdot 2) (2 \cdot 5)\) to make the task of computing the product easier. (a) Did this informal strategy implicitly entail the associative property of multiplication, the commutative property of multiplication, or both? (b) Explain your answer.

4. What meaning could be attached to expressions such as \(? = 5 + 3\) and \(? = 2 + 8\)?

5. (a) Like other operations with whole numbers, division can be viewed as a function. Illustrate how this could be shown using an In-Out Machine. (b) Consider all the possible outputs for such a machine for the whole numbers 1 to 12. In what way will the outputs for the machine differ from those that represent addition, subtraction, and multiplication?

6. Consider how you would help children see which operations had the commutative and associative properties and which did not.

7. Given 14 - 7, Deke, a first grader, commented, "That's [four after ten and that's] three before ten, so it's seven." This appears to be an abbreviated verbal description of an interesting thinking strategy. Briefly, describe the thinking strategy step by step.

**SUGGESTED ACTIVITIES**

1. Make a list of everyday situations in school that could provide the basis for addition, subtraction, multiplication, and division problems. Classify the situations according to the main meanings of each operation described in the *Student Guide*.

2. The wording of story problems can have a profound effect on children's performance. For example, with part-part-whole unknown-part problems, kindergartners and first-graders do better if the problem includes wording that clarifies the relationship between the parts and the whole (DeCorte, Verschaffel, & DeWin, 1985). The italicized portion added to the problem below accomplishes this purpose.

   (Text continued on page 125.)
Mathematics instruction should take into account what we know about children’s mathematical learning. Answer the following questions by yourself or with your group. Share your ideas with your group or class.

1. *The Math Book From Hell* (3rd-grade edition) introduces multiplication in the following manner:

   - The first page of the multiplication chapter introduces the operation with examples such as \(5 + 5 + 5 = 3 \times 5, 2 + 2 + 2 + 2 = 4 \times 2, 3 + 3 = \underline{4} \times 3\).
   - The first page proceeds to define multiplication as “an operation on two or more numbers called factors to find a product” and illustrates the definition with the following example:

\[
\begin{array}{c}
\text{factor} \\
\downarrow \\
4
\end{array} \times \begin{array}{c}
\text{factor} \\
\downarrow \\
5
\end{array} = \begin{array}{c}
\text{product} \\
\downarrow \\
20
\end{array}
\]

   (a) The examples used by the textbook imply what meaning of multiplication? (b) Does the textbook use a developmentally appropriate approach to introduce multiplication? Why or why not? (c) How helpful do you think its definition of multiplication is?

2. The second page of the introductory chapter on multiplication contained a sample equation \(3 \times 2 = \_\) and the following instructions: “Find each product.” The first row of items and a child’s answers are shown below:

\[
\begin{array}{cccc}
2 \times 1 &=& 8 \\
2 \times 2 &=& 4 \\
2 \times 3 &=& 5 \\
2 \times 4 &=& 6 \\
2 \times 5 &=& 7
\end{array}
\]

   (a) What common systematic error is the child making? (b) Why may the child be making this error?

3. The third-grade edition of *The Math Book from Hell* has three units on multiplication in all. The first chapter begins with a chapter on the times-two and times-three families. The next chapter focuses on times-four and times-five combination. The third chapter focuses on times 0 and times 1. It explains that there is a shortcut for learning these facts and spells out the \(n \times 0 = 0\) and \(n \times 1 = n\) rules. The second unit begins with a chapter on times-six and times-seven facts. The next chapter focuses on times-eight and times-nine. The third chapter reviews all the basic (single-digit) combinations. A number-fact mastery test is included at the end of the chapter. Later, the third unit focuses on multiples of 10.

   a. Based on what is known about children’s informal knowledge of multiplication, evaluate the structure of instruction just outlined. In particular, does the order of the chapters make sense? With what combination would it make the most sense to introduce multiplication? Why? Does it make sense to delay the introduction of combinations involving 0 and 1, or should these combinations be introduced from the start?

   b. Is memorizing \(n \times 0 = 0\) and \(n \times 1 = n\) rules by rote likely to be effective? Why or why not?

   c. The textbook implies that instruction should jump quickly into having children memorize the basic number combinations. Is this developmentally sound?

4. *The Math Book from Hell* introduces the area and the combinations meanings of multiplication as enrichment activities. Is this likely to be sufficient to give children a firm and broad understanding of multiplication?
Investigation 5.A: Using Concept Maps and Inductive Reasoning to Explore Division

◆ Reflecting on the connections among division concepts ◆ 5-8
◆ Whole class, small groups, or individually

Part I: Constructing a Concept Map

Add the following concepts to the concept map begun in Investigation 5.1 (page 5-10 of the Student Guide): area meaning, division, divvy-up meaning, and measure-out meaning. Specify clearly the links among these concepts and their links with the addition, subtraction, and multiplication concepts and arithmetic principles already on the map.

Part II: Evaluating a Concept Map

Rafi, a seventh-grader, drew the concept map of division shown on the next page.

1. What relationships with other operations did he overlook?

2. (a) Are Rafi’s labels of the elements (Smaller sets and Larger Sets) in the measure-out and divvy-up meanings accurate for whole-number division (where the quotient is a whole number)? Why or why not? (b) For numbers in general, what mathematically more correct labels should he use? Why?

3. (a) Complete the representation of the Groups-of meaning for multiplication and show how each element of this meaning, including the answer (product), is related to the elements of a measure-out and a divvy-up meaning. (b) Write an example of a groups-of multiplication problem. (c) Rewrite the problem you created for Question 3b as a missing-factor problem so that it embodies a divvy-up meaning. Do the same for a measure-out meaning.

Part III: Inducing the Properties of Division

1. Division has what properties? Test each statement below with several examples by substituting whole numbers for the shapes (e.g., let a triangle, box, and circle = 8, 4, and 2, respectively, or 2, 6, and 12, respectively). Recall that it takes only one counterexample to reject a conjecture. Circle the letter of relevant properties; put an x through the letter of irrelevant ones (those for which you find a counterexample). Discuss your results with your group or class.

a. Associative:

b. Commutative:

c. Continuous:

d. Divisor-Quotient:

e. Distributive-Like:

2. (a) Use letters to summarize algebraically the expressions in the previous question. Use an = in your algebraic expression to indicate that property is applicable to division; use an ≠ to indicate that the property is not applicable to division. (b) Did Rafi represent the laws of division accurately in his concept map on the next page?

* Note that concept map on the next page and inducing all the properties above might be appropriate for seventh- or eighth-graders. Fifth- or sixth-graders could be asked to evaluate a simpler concept map and to discover whether or not division has the associative and the commutative properties.
Investigation 5.A continued
Problem 5.A: An Improved Part-Part-Whole, Unknown-Part Word Problem (u 1-3). Six children helped the teacher clean up. Four were boys, and the rest were girls. How many girls helped the teacher?

The wording of compare word problems can also have a significant effect on children's ability to understand such problems (Hudson, 1983). Below is a standard compare problem:

Problem 5.B: A Standard Compare Problem (u 2-4). There are six birds and four worms. How many more birds than worms are there?

A modified type of compare problem, such as Problem 5.C below, is easier for children to understand and to solve (Fuson, 1992):

Problem 5.C: A "Won't-Get" Compare Problem (u 2-4). There are six birds and four worms. If no bird takes more than one worm, how many birds won't get a worm?

(a) Find two part-part-whole unknown-part problems in a textbook and rewrite them to clarify the relationship between the parts and the whole. (b) Find two standard compare problems in a textbook and covert them into won’t-get compare problems.

Illustrated below is a sample Home-Connections activity. Devise another that helps children connect single-digit arithmetic to their everyday life.

Dear Parent or Guardian,

I have asked your child to keep a journal for one full week on how he or she uses addition and subtraction outside of school. Examples can be illustrated with hand-drawn pictures or pictures cut from a newspaper, magazine, or catalogue. Would you please help your child reflect on the use of addition and subtraction in your home as your child prepares the journal. Thank you.

(a) Make a list of board and computer games that children could play with their parents, siblings, or friends at home to practice basic arithmetic skills. (b) Analyze the games to identify which specific arithmetic skills (or concepts) each involves. Justify your decisions. (c) Create an index on 3 x 5 cards or in a computer file.

Check to see how multiplication is defined by a textbook series in its third- and fourth-grade volumes. Consider whether the definition would make sense to children (e.g., builds on their informal knowledge of multiplication). How many different meanings of multiplication does the textbook series introduce? Compare your findings with those of others who examined different textbook series.

Evaluate several textbook series in terms of the guidelines for teaching basic number combinations on page 5-27 to 5-33 of the Student Guide. Does each encourage a counting and a reasoning phase and building on existing knowledge (e.g., helping children master 2 x n combinations by building on their knowledge of the addition doubles)?

Tutor a first grader on addition or subtraction or a third grader on multiplication or division for a period of time prescribed by your instructor. (a) Identify the child’s informal and formal strengths and weaknesses. (b) Delineate specific goals for the child, and how s/he can learn about the operation in a meaningful manner. (c) Assess the child’s progress, and evaluate the effectiveness of your instructional plan. (d) Develop a list of recommendations for further instruction.

For a period of time prescribed by your instructor, tutor an elementary school child who is having difficulty mastering the basic addition, subtraction, multiplication, or division combinations. (a) Identify informal and formal knowledge that you can build on. (b) Devise an instructional plan for the counting phase (if needed), the reasoning phase, and mastery phase. Include in the plan specific activities, problems, games, projects, and so forth that will be used for purposeful instruc-
tion and practice. (c) Assess the child’s progress, and evaluate the effectiveness of your instruction. (d) Develop a list of recommendations for further instruction.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. Use the Taxonomy of Addition and Subtraction Word Problems (Figure 5.2) on page 5-4 of the *Student Guide* to identify the type of problem and the missing term for each of the following word problems. Fill in the blank next to each problem with the letter below that correctly identifies the problem type.

   - A. Change Add To: Unknown Outcome
   - B. Change Add To: Unknown Change
   - C. Change Add To: Unknown Start
   - D. Change Take Away: Unknown Outcome
   - E. Change Take Away: Unknown Change
   - F. Change Take Away: Unknown Start
   - G. Part-Part-Whole: Unknown All or Whole
   - H. Part-Part-Whole: Unknown Part
   - I. Compare: Unknown Difference
   - J. Compare: Unknown Part
   - K. Equalize: Unknown Difference
   - L. Equalize: Unknown Part

   **a.** Arkady had 12 points but then lost 2 points. How many points does he have now?

   **b.** Miska had 3 points and then scored 4 more. What is Miska’s new total?

   **c.** Arkady had 12 points and Miska had 7 points. Miska needs to score how many more points to catch up to Arkady?

   **d.** When the game ended, Arkady had 12 points and Miska had 9. By how many points did Miska lose?

   **e.** Arkady had 11 marbles in a bag. Miska put some more marbles in Arkady’s bag. Arkady counted the marbles in his bag and found there were now 13. How many marbles had Miska given him?

   **f.** Arkady had 11 marbles. Seven were clear and the rest were cat eyes. How many cat eyes did he have?

   **g.** Nina made 8 decorations and gave most of them to friends. She had 2 left. How many did she give away?

   **h.** Nina made 8 decorations. Sue made 3 fewer. How many did Sue make?

   **i.** Nina made 8 decorations. Sue would have to make 3 more to have the same number. How many did Sue make?

2. For each of the first four problems in Question 1 above (a, b, c, and d) indicate whether children would tend to use a (A) concrete counting-all, (B) concrete take-away, (C) concrete equalizing, or (D) matching strategy to concretely model the problem.

3. Circle the letter of any of the following statements that—according to the *Student Guide*—are true. Change the underlined portion of any false statement to make it true.

   - a. Ms. Stress explains that it is essential that first graders not get into the bad habit of counting. She advocates moving immediately to drilling the basic facts so that children can memorize them quickly. Research supports Ms. Stress’ view.

   - b. Children typically can interpret and solve change add-to missing-outcome problems before they can interpret and answer formal expressions such as $5 + 3 = \square$.

   - c. Word problems, rather than symbolic expressions, should be used to introduce arithmetic operations with whole numbers.

   - d. Ms. Stress wanted to evaluate whether or not her students really understood symbolic addition. Miss Joiner recommended giving the students a symbolic expression like $5 + 3 = ?$ and asking them to write a word problem for it. Research supports Miss Joiner’s recommendation.

   - e. Ms. Stress did not think it was a good idea to introduce first graders to a wide variety of addition and subtraction problems. She recommended postponing the introduction of compare, equalize, and missing-addend (e.g., missing-change) problems to 3rd or even 4th grade when children might be developmentally more ready.
Research, including cross-cultural research, supports Ms. Stress' reservations.

f. Young children in traditional programs tend to view \(5 + 2 = 7, 7 = 7, 7 = 5 + 2,\) and \(5 + 2 = 8 - 1\) as "correct" and "sensible."

g. Children in traditional programs tend to interpret the equals sign as meaning "sums to" or "makes."

h. Formally, the equals sign has an operator meaning.

i. To encourage a formally correct view of the \(=\) sign, a teacher should refer to it as "equals."

j. By the end of kindergarten, many children understand that the subtraction of one can be undone by the addition of one and vice versa.

k. At first, children frequently do not realize that commuted problems, such as \(7 + 3\) and \(3 + 7\), have the same sum because of their informal "part-part-whole" view of addition.

l. Initially, it may be easier for children to determine the sum of \(n + 1\) expressions such as \(8 + 1\) than \(1 + n\) expressions such as \(1 + 8\) because of their informal concept of addition.

m. Because they are similar and easily confused, a teacher should not introduce equalize and compare problems together.

n. Missing-addend equations such as \(5 + ? = 8\) should be introduced with problems such as change add-to missing change problems.

o. The inverse principle of addition and subtraction can be represented by the equation \(9 - 3 = 3 + 3\).

p. For meaningful learning, children should master arithemtic concepts in an informal, concrete manner first and then work with the formal, written symbolism.

q. Because of their informal knowledge, children tend to view the equation \(7 - 3 = 4\) as "seven take away three leaves four" or "the difference between seven and three is four" but not "four is how much more must be added to three to make seven."

r. Children do not initially recognize that different combinations such as \(5 + 3\) and \(4 + 4\) can have the same sum.

4. In *Young Children Reinvent Arithmetic*, Constance Kamii (1985) noted that "teachers all over the world know that questions such as \(2 + \square = 6\) and \(\square + 4 = 6\) are hard for first graders." She concluded that there is no point in introducing missing-addend questions to first-graders. Is this position consistent or inconsistent with that taken by the text? Briefly justify your answer.

5. Analyze each of the following cases. (a) Use the information in the *Student Guide* to outline a general plan of how a teacher could help the child described in each case. (b) Indicate what specific word problems, concrete activities, games, and so forth you could use to implement your general plan.

**Case 1.** Brita, a first grader, does not appear to understand formal subtraction symbolism that had been introduced during her prolonged absence from school. For example, for \(4 - 1\) and \(5 - 3\), she recorded answers of 3 and 2, respectively.

**Case 2.** Flavius, a first grader, does not understand the rationale for the subtraction checking procedure. As a result, he mechanically adds the difference to the subtrahend and proceeds to the next item whether or not his answer "checks out."

**Case 3.** Neola, a second grader, has just transferred from another school and does not understand missing-addend notation. For example, for \(5 + \square = 8\), she answered 13 and, for \(7 + \square = 9\), she answered 16.

**Case 4.** Unlike her peers who have invented a relatively efficient counting-on procedure, Asha has nearly completed first grade but still relies on the relatively inefficient strategy of counting-all to determine sums.

**Case 5.** Aba, a second grader, has no difficulty solving change take-away problems, determining the difference to a symbolic expression such as \(7 - 4\), or translating take-away problems into symbolic expressions. However,
when given a "difference" problem, such as that in Question 1d on page 126, and asked to represent it and its solution as an equation, she did not know what to do.

**Case 6.** Mrs. Mohed found that Ezzie and many of his classmates did not have a mathematically accurate understanding of the equals sign. As a result, they did not understand the algebraic principle that what is done to one side of the equation must be done to the other (the balance principle).

**Case 7.** Shadwell seems confused about multiplying with zero and one. He regularly makes errors such as $7 \times 0 = 7$, $0 \times 3 = 3$, $6 \times 1 = 7$ and $1 \times 4 = 5$.

**Case 8.** Hobart can’t remember whether dividing by 0 or 0 divided by a number is indeterminant.

6. During student teaching, you notice that students do not understand the add-back procedure for checking subtraction (e.g., $43 - 27 = 16 \rightarrow 27 + 16 = 43$). (a) What principle provides a rationale for the checking procedure? (b) How can you help children understand why this procedure works?

7. The *Mathematics Their Way* activity of using green and red colored beans to show different ways of making five (4 green and 1 red, 2 green and 3 red, and so forth), for example, fosters the development of what concepts? Briefly justify.

8. Playing the game *Fill In* (Activity File 5.2 on page 5-13 of the Student Guide) is useful for fostering which of the following concepts?

   a. associative principle
   b. commutative principle
   c. inverse principle
   d. missing-part concept
   e. other-name-for-a-number concept
   f. part-part-whole concept
   g. relational view of equals
   h. same-sum-as concept
   i. subtraction (e.g., $5 - 3 = \square$) is related to addition (e.g., $3 + \square = 5$)

9. (a) For each multiplication word problem below, identify the multiplication meaning by filling in the blank with the appropriate letter: $A =$ groups-of, $B =$ rectangular array, $C =$ rate, $D =$ combinations, $E =$ area, and $F =$ comparison. (b) Put a check (√) next to those that represent an asymmetrical meaning.

   - a. Arnie makes $5 for each lawn he cut. He cut six lawns. How much did he make?
   - b. Arnold had four paint colors and three different types of symbols with which to decorate his model airplane. If he chose one color and one symbol, how many different ways could he decorate his model airplane?
   - c. Arnold had six model airplanes. His brother had three times as many. How many models did his brother have?
   - d. Avi dug up three plots each 12 square-feet in area. What was the total area he dug up?
   - e. Alistair saw 3 repetitions of 5 flashes from the forward observer. If each flash indicated an enemy soldier, how many enemy soldiers had been spotted by the forward observer?
   - f. Mr. Tinsley the grocer had just stacked bags of Dainty Dog dry dog food when Ruffus bolted from the family car, charged into the store through an open door, and headed directly to the pet-food aisle. Before Mary or her parents could stop the run-away dog, Ruffus had bit or clawed every one of the bags of Dainty Dog dry dog food. If Mr. Tinsley had stacked the bags 5 rows high with 4 bags in a row, how many bags of dry dog food did Mary’s family have to buy because Ruffus had ruined them?
   - g. Akili had to paint a wall 8 feet long by 6 feet high. How many square feet of wall did he have to paint?

10. Circle the letter of any of the following statements that is, according to this guide, true. Change the underlined portion of any false statement to make it true.

   a. Rectangular array, area, and comparison situations are symmetrical and, hence, neither factor is clearly the multiplier.

   b. A repeated-addition meaning is important for understanding fraction and decimal multiplication later.
c. Psychologically, it makes sense to begin multiplication instruction with the times-zero, times-one, and times-two—the smallest times combinations, because children can readily use a groups-of interpretation and their informal knowledge to figure out the products of such combinations.

d. Division can be introduced informally in the contexts of fair sharing.

e. A missing-multiplier (number-of-groups) multiplication problem is related to a measure-out meaning of division.

f. The following is a measure-out division word problem: If Bridget has 12 candies and puts 2 candies in each bag, how many bags can she fill?

g. The following is a divvy-up word problem: Bridget has 12 candies. She has four bags for each of her friends. How many candies can she put in the bag?

h. The following is a divvy-up word problem: Bridget has 12 candies. She eats three a day. How many days will the candy last?

i. The following word problem is a measure-out problem: Janice picked 1000 apples and bagged them. Each bag contained 25 apples. How many bags did Janice need?

j. 0 ÷ 4 is indeterminant, and the quotient of 4 ÷ 0 is zero. (Consider the memory aids suggested by the Student Guide for helping children to remember this.)

11. Indicate whether the following models are more appropriate for a divvy-up problem (write D) or a measure-out problem (write M).

   a. Repeated subtraction such as 12 ÷ 3: 12 - 3 = 9 (1), 9 - 3 = 6 (2), 6 - 3 = 3 (3), 3 - 3 = 0 (4); answer = 4.

   b. Figuring out 12 ÷ 3 with objects by counting out 12 items, making groups of three until the 12 items are used up, and then counting the number of groups to determine the answer 4.

   c. Figuring out 12 ÷ 3 with tallies by placing a tally in three circles until 12 tallies have been made and then counting the number of tallies in each circle to determine the answer 4.

12. Base-ten blocks consist of: small cubes (ones), longs (tens), flats (100s) and large cubes (1,000s). (a) Demonstrate or illustrate how base-ten blocks can be used to represent a groups-of interpretation of the expression 4 x 5. (b) Demonstrate or illustrate how they could be used to represent an area meaning of 4 x 5.

13. (a) Demonstrate or illustrate two ways discussed that you could pictorially illustrate 3 x 0 to convince a third grader that the product is zero? (b) Many students in your class are confused by division involving 0. Edger, for example, believes that both 0 ÷ 4 and 4 ÷ 0 are 0. Describe three different ways you could help your class and Edger, in particular, understand the difference between dividing zero by a number and dividing a number by zero.

14. Demonstrate or illustrate how base-ten blocks could be used to represent (a) a divvy-up meaning of 20 ÷ 4, (b) a measure-out meaning of 20 ÷ 5, and (c) an area meaning of 20 ÷ 5.

15. Circle the letter of any of the following statements that is, according to the Student Guide, true.

   a. Children typically master the basic number combinations gradually.

   b. The basic number combinations provide very few opportunities to search for patterns or relationships.

   c. Thinking strategies such as 'doubles plus one' typically have to be taught to children because they seldom invent such sophisticated procedures themselves.

   d. Mastering the basic number facts should focus on repeated practice.

   e. Subtraction combinations like 7 - 3 = 4, 5 - 2 = 3, 8 - 6 = 2, and 6 - 4 = 2 typically are not affected by the level of children's mastery of addition combinations.
f. Adults use facts and relationships stored in long-term memory to efficiently produce basic number combinations, such as $7 + 0 = 7$, $8 - 7 = 1$, $5 \times 4 = 20$, and $20 \div 4 = 5$.

g. Your first-grade class focuses on solving word problems, while Mr. Linn’s class focuses on the drill and practice of basic skills. On an achievement test, you can expect your children to do better on the problem-solving section, the concepts section, and knowledge of basic arithmetic facts section.

16. At mid year in the first grade, Gloria still used counting all to compute sums. She even had to used this strategy to compute the sums for combinations involving one (e.g., $6 + 1$ and $1 + 8$). (a) How might Gloria’s teacher encourage a counting-on strategy? (That is, what specific method or methods might the teacher use to foster the invention of this more advanced counting strategy for determining sums?) (b) What could be done to help Gloria master the combinations involving one?

17. In each of the following cases, indicate how a teacher could encourage the child to build on what he or she already knows to devise an appropriate thinking strategy.

   a. Attica did not even know the addition-with-one combinations such as $8 + 1$, $1 + 6$, $1 + 4$, $9 + 1$, and $7 + 1$.

   b. Bettina had mastered the combinations involving one and the doubles such as $3 + 3 = 6$ but still counted out the sums for expressions such as $4 + 5$, $8 + 7$, $6 + 7$, $4 + 3$, $3 + 2$, and $5 + 6$.

   c. Cassidy had special difficulty remembering combinations involving nine such as $9 + 6$, $4 + 9$, $9 + 7$, and $5 + 9$.

   d. Dora had learned some of the basic subtraction combinations but still had to compute the difference for combinations such as $5 - 4$, $7 - 6$, $8 - 7$, and $9 - 8$.

   e. Elan typically had to compute the difference of miscellaneous basic subtraction combinations such as $7 - 4$, $13 - 8$, $14 - 6$, $13 - 7$, $12 - 9$, $11 - 7$, $10 - 6$, and $9 - 5$.

f. Fernando a third grader still calculated $2 \times n$ combinations such as $2 \times 4$ and $2 \times 7$ by counting by twos.

g. Giordano had difficulty learning basic division combinations such as $8 \div 4$, $16 \div 2$, $27 \div 3$, $36 \div 9$, and $56 \div 7$.

18. Specify what prerequisite knowledge children would need to build on in order to devise an appropriate thinking strategy for (a) combinations involving the addition of one, (b) combinations involving the subtraction of one, (c) basic combinations involving the addition of 9, (d) miscellaneous basic subtraction combinations, (e) times-two combinations, and (f) miscellaneous basic division combinations.

19. Ms. Stress ensured that her class had a wealth of practice on the basic number combinations. She had them do a worksheet and flash cards each day. Each week she had timed tests and a number-fact “bee” (contest). Miss Joiner had her class focus on solving word problems and playing math games. The results of the research program titled Cognitively Guided Instruction (CGI) can best be summarized by which one of the following statements?

   a. Miss Joiner’s class should outperform Ms. Stress’ class on number-combination mastery.

   b. Miss Joiner’s class should do about as well as Ms. Stress’ class on number-combination mastery.

   c. Miss Joiner’s class should do significantly less well as Ms. Stress’ class on number-combination mastery but should outperform them on a problem-solving test.

   d. The CGI results are not relevant to this situation.

20. McCloskey, Caramazza, and Basili (1985) reported on a case study of an adult who used the following strategy to determine the quotient of $56 \div 7$: The divisor (7) was multiplied by 10. Then the dividend was subtracted from the product (70). Next, the result (14) was divided by the divisor. The result of this last operation (2) was subtracted from 10 to obtain the correct answer (8). Prove why this strategy works.
WRITING OR JOURNAL ASSIGNMENTS

1. A fellow student is confused about the difference between change add-to and part-part-whole problems. What could you point out to help the student distinguish between these two types of addition problems?

2. Change add-to and part-part-whole problems are considered addition problems, whereas change take-away, equalize, and compare are considered subtraction problems. Does this classification refer to how such problems are solved (what operation is used to determine the answer)? Why or why not?

3. In the Mathematics Their Way Summary Newsletter, Garland (1988) noted that "before children can successfully solve a word problem, they must be able to . . . choose the number operation(s) necessary to solve the problem . . ." (page 10.4). For this reason, word problems are not used to introduce addition and subtraction. Evaluate Garland’s comment in light of the research on how young children solve word problems.

4. Consider each of the following vignettes in light of children’s informal change add-to view of addition and then answer the questions that accompany each vignette.

a. Vignette I: An Interesting Discrepancy. Presented with the symbolic expression $5 + 1$ and asked, What is the sum of five plus one? Reuben, a kindergartner, simply shrugged his shoulders. Asked How much is five candies and one more candy? he promptly responded, "Oh, that’s six." Why was Reuben unable to answer the first question, yet had no difficulty answering the second?

b. Vignette II: Informally Discovering the Commutative Property of Addition. The first time Kate, a kindergartner, was shown a pair of commuted addition expressions (2 + 4 and 4 + 2) one immediately below the other, she did not indicate whether the two would add up to the same or different answers. (She had already quickly indicated that identical expressions such as $6 + 1$ and $6 + 1$ would and that expressions with a different addend such as $2 + 5$ and $2 + 10$ would not.) Later, when asked about $5 + 3$ and $3 + 5$, she responded, "I can't tell." Two trials later when asked about $4 + 6$ and $6 + 4$, the girl noted, "That’s the one I got so much trouble over. Same?" A week later, Kate was reinterviewed. This time she responded to the six pairs of commuted expressions by quickly indicating, "The same." Asked to explain her response, Kate noted, "Because same numbers in different places look like they add up to the same." In a third session, Kate had to compute the sum of $6 + 4$ but when shown $4 + 6$ next, quickly answered, "Ten." Asked to justify her response, Kate explained, "I figured it out when I counted when we played the other game." In brief, by informally computing the sums of several pairs of commuted expressions, Kate had discovered an important property about addition—one that she ultimately could apply consistently and with confidence. (i) How might Kate's informal view of addition have prevented her from understanding the commutative property in the initial interview? (ii) How might children discover the commutative property of addition in an even more concrete manner than did Kate?

c. Vignette III: Equal Sign Confusion. Many children seem to have difficulty making sense of expressions in formats other than that traditionally taught (e.g., $8 + 3 = 11$). For instance, asked if $8 = 8$ was correct, Madison responded, "No, you forgot something. If you put plus zero $[8 + 0 = 8]$, it would be okay." Likewise, the child claimed that the expression $7 + 3 = 3 + 7$, which illustrates the principle of commutativity, and expressions such as $10 = 3 + 7$, $7 + 3 = 5 + 5$, and $3 + 7 = 12 - 2$, which illustrate the more general other-names-for-a-number concept were incorrect. Why are such formats so confusing to this and many other children?

d. Vignette IV: Missing-Addend "Problem." Many primary-level children have great difficulty with missing-addend equations. August, for instance, responded to $7 + ? = 10$ with an answer of
seventeen.  (i) What might account for this response?  (ii) How could missing-addend expressions be introduced to young children in a meaningful fashion?

5. You notice that your first graders can informally compute the sum of expressions such as \(8 + 1, 7 + 3, 9 + 2\) but have great difficulty doing so with expressions such as \(1 + 8, 3 + 7,\) and \(2 + 9\). Briefly explain why. Your answer should include the role played by children's informal view of addition.

6. Write a word problem that involves (a) a groups-of meaning of multiplication, (b) an area meaning of multiplication, and (c) a comparison meaning of multiplication.

7. You want your class to be able to state the products of the multiplication combinations involving two quickly and accurately. You examine a textbook, which gives the following guidance:

"Give a few \(\times 2\) problems to which the whole class responds in unison. Then do several more in which the students respond with response cards or write the answers on papers as you walk around the room checking. Continue quizzesing them on the \(\times 2\) facts . . . (Willoughby, Beren-iter, Hilton, & Rubinstein, 1987).

The textbook goes on to note that if children still have not mastered combinations with at least one factor involving 2, then have them work in pairs or small groups so that they can quiz and help each other. Evaluate these guidelines for helping children to master the basic multiplication combinations involving 2. More specifically, note strengths and weaknesses about the recommendation. Does the prescribed approach build effectively on what children already know? (That is, is it consistent with the recommendation made by the Student Guide?) Why or why not?

8. Your first teaching assignment during student teaching is to introduce the class to symbolic division, including that involving a remainder notation (e.g., \(9 \div 4 = 2 \text{ r} 1\)). Briefly explain how you would attempt to do this.

9. Arvin doesn't understand what \(\text{r}2\) in the equation \(14 \div 4 = 3 \text{ r} 2\) means. How could you help Arvin understand this symbolism?

10. Might relating formal division notation such as \(14 \div 4 = 3 \text{ r} 2\) to an area model be helpful? Why or why not?

11. (a) When is \(2 + 2\) not equal to 4? (b) When is \(1 + 1\) not equal to 2?

**PROBLEMS**

■ **A Number Riddle** (◆ 3-8)

Which pair of numbers summing to 24 has the largest product?

■ **A Nonroutine Division Problem** (◆ 3-8)

The following problem is similar to a problem found in a third-grade textbook: Enrique had some marbles. He divided them into two piles and gave one to Frederico. Frederico divided his pile into two piles and gave one pile to Georgio. Georgio had 4 marbles. How many marbles did Enrique have to begin with? (a) Why is this word problem ambiguous? (b) What answer was probably intended? (c) As written, what is the answer to the problem?

■ **Calculated Remainder**† (◆ 6-8)

Using the \(\div\) key, the Texas Instruments Math Explorer—like any typical calculator—will specify the results of division as a decimal (e.g., \(13 \div 4 = 3.25\)). Using the \([\text{INT} \div]\) key (integer division key), it—unlike typical calculators—will specify the results of division as an integer and a remainder (e.g., \(13 \div 4 = 3 \text{ r} 1\)). Assume you do not have a calculator with an \([\text{INT} \div]\) key. Using just the basic operation keys (+, -, \(\times\), and ÷), how could you use such a calculator to determine the remainder of 48,936,725 \(\div 9,673\). Justify why your procedure works. What is the remainder?

■ **Brother, Can You Spare a Tire?** (◆ 7-8)

A four-wheel car is equipped with tires that will last exactly 20,000 miles. What is the least number of spare tires that must be carried so that a marathon race of exactly 30,000 miles can just be completed? Describe how the tires would be used.

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† Based on a problem described in "Prospective Elementary Teachers' Knowledge of Division" by Martin A. Simon, appearing in *Journal for Research in Mathematics Education*, 1993 (Vol. 24), pp. 233-254.
ANSWER KEY for Student Guide

Key for Probe 5.1 (pages 5-6 and 5-7)

Part I
1. Problem 1 = change take-away, missing change; Problem 2 = change add-to, missing change; Problem 3 = part-part-whole, missing whole; Problem 4 = compare, missing difference; Problem 5 = equalize, missing difference; Problem 6 = change add-to, missing change; Problem 7 = part-part-whole, missing part.

2. (a) Change add-to or take-away problems (Problems 1 and 2) involve situations that begin with a single collection and something that changes this amount. More specifically, change add-to problems entail adding something more to the initial collection to make it larger. Change take-away problems entail removing something from the initial collection to make it smaller. (b) Part-part-whole problems (Problem 3) and the two types of difference problems (equalize and compare problems; Problems 4 & 5) begin with two numbers, which are either added or subtracted to find the whole, the difference, or one of the parts.

3. (a) Change problems cue action: adding or removing something. Equalize problems likewise imply action: an effort to add (or remove) items from a collection to make it the same as another collection. Problems 1, 2, 5, & 6. (b) Unlike “active” problems, part-part-whole and compare problems are “static” in that they do not imply action. For example, unlike equalize problems, compare problems imply that the difference between the two collections will persist. The operation (adding or subtracting) is done merely to determine the extent of the difference, not eliminate it as with equalize problems. Problems 3, 4, & 7.

Part II
2. a. Concrete counting-all: Problems 2 and 3 b. Concrete taking away: Problem 1 c. Concrete equalizing: Problem 5 d. Matching: Problem 4


Part III
1. (a) Task C involves a concrete counting strategy and thus should be the easiest. (b) Task A involves symbolic addition, which is relatively difficult for young children.

2. (a) MTW is consistent with a developmental approach described in the Student Guide in that it introduces word problems before symbolic addition and subtraction. (b) It is inconsistent in that word problems are not used to introduce the operation. Note, though, that preschoolers can solve concrete problems before word problems (e.g., Huttenlocher, Jordan, & Levine, 1994).

3. (a) Change take-away. (b) The difference (compare and equalize) meanings.

Key for Investigation 5.1: Questions for Reflection (page 5-10)

1. To make the task more manageable for primary-level students, begin with relatively few concepts and links. For example, first graders might begin with just the following three concepts: addition, change add to (unknown outcome) and part-part-whole (unknown whole). Later they could make a concept map of subtraction and its meanings. In time, these two maps could be combined and elaborated on.

2. A concept map could include the following links: addition and subtraction are inverse operations—i.e., one operation undoes the other (e.g., 5 - 2 + 2 = 5), addition can be used to check subtraction, and knowledge of addition combinations can be used to determine
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differences (e.g., \(5 - 3 = ?\) can be thought of as \(3 + ? = 5\)).

3. The commutative principle specifies that the order of the addends does not affect the outcome. Some students incorrectly believe that this principle applies only to situations involving two addends. To help students rethink this assumption, encourage the class to offer a counterexample or to consider which principle applies to the following situation: \(5 + 3 + 5 = 5 + 3 + 5\). Because the order of the addends has changed, the commutative principle applies. Associativity is concerned with grouping. In the following example, note that the grouping of the addends has changed but the order of the addends has not \((5 + 3) + 5 = 5 + (3 + 5)\).

4. (a) A traditional skills approach typically fails to help children understand the rationale for "add-back" method for checking subtraction because it fails to connect the procedure to children’s informal knowledge of the addition-subtraction inverse principle—that addition and subtraction can undo each other. For example, if three is subtracted from five, this can be undone by adding three back. That is, adding three cancels taking three away, which leaves you the number you started with. (b) Demonstrating the inverse relationship with a small number of items can be helpful.

Key for Probe 5.2 (page 5-15)

Part I

1. D 2. E 3. A or C 4. A or C (Note that mention of the word area does not automatically make a problem an area problem.) 5. B 6. F

Key for Probe 5.3 (page 5-16)

1. (A) Groups of. (B), (C), and (D) Groups of. (They could also be viewed as rate models or possibly comparison models.) (E) Rectangular array. (F) Rectangular array. (G) Crossing points is typically defined in mathematics-education textbooks as a combination model (e.g., If you have 4 vertical strings and 3 horizontal strings, how many different knots can made where they cross?). Many students view crossing points as a rectangular array model (as points in a row), which seems reasonable. (H) Combinations. (I) Combinations. (J) Rectangular array. (It would be an area model if you counted the squares of the graph paper rather than the dots.) (K) Combinations. (L) Area or rectangular array depending on how the manipulative is conceptualized. (M) Area.

2. Example B illustrates a rectangular-array model of 1 x 12, not 3 x 4. A rectangular-array model of 3 x 4 would involve three rows of four interlocking blocks each.

3. (a) Models A and H can be helpful in explaining why a number times zero is zero. For example, \(3 \times 0\) could be represented as three empty circles (Model A) or three horizontal lines and no crossing points (Model G). (b) Model A and G can also be helpful in explaining why zero times a number is zero. For instance, \(0 \times 3\) could be represented as no circles of three (Model A) or three vertical lines and no crossing points (Model G).

Key for Investigation 5.2: Questions for Reflection (page 5-19)

1. (a) Problem H is a divvy-up problem. (b) An informal strategy for modeling and solving this divvying-up problem is a divvying-up strategy: (i) Represent the amount (32 candies) by counting out 32 items, (ii) deal out an item to each of four piles (which represent the four children), (iii) continue this dealing-out process until all 32 items have been distributed among the four piles, and (iv) count the number of items in a pile to determine the size of a child’s share. (c) Problem I is a measure-out problem. (d) An informal strategy for modeling and solving this measure-out problem is a measure-out strategy: (i) Count out 32 items to represent the total amount, (ii) make a subgroup of four to represent the size of one share, (iii) repeat Step 2 until all 32 items are used, (iv) count the number of subgroups of four to determine the number of shares.

2. A number-line representation would be appropriate for Problem I, a measure-out problem.

3. Repeated subtraction is most closely associated with measure-out problems, such as Problem I.
Part I

1. Research indicates that adults use (a) fact recall, (b) reasoning, and even (c) counting to quickly and accurately determine the answers to basic number combinations (e.g., Browne, 1906; LeFevre et al., 1996a, 1996b).

2. For both 5 + 1 and 4366 + 1, for example, experts might draw on their counting knowledge and the number-after rule for adding one to answer quickly and accurately.

Part III

There are numerous patterns in a table of the 100 basic multiplication combinations. A class could easily spend a period or more looking for and discussing them. A number of patterns that could be used in the reasoning phase and in the meaningful memorization of single-digit multiplication combinations are discussed in Box 5.6 on pages 5-31 to 5-33 of the Student Guide. The first four patterns listed below could also be helpful in these endeavors.

1. Rows and columns entail skip counts.

2. Even \( x \) even = even; odd \( x \) even = even; and odd \( x \) odd = odd. Children may recognize special cases of these general rules including the fact that all \( 2 \times n \) and \( n \times 2 \) are even numbers, multiplication of five and an even number yields a product with a 0 in the ones place, and multiplication of five and an odd number yields a product with a five in the ones place.

3. Products of times-nines combinations form simple arithmetic progressions: The tens-place digit increases by one each time, and the ones-place digit decreases by one each time. Moreover, in the list of \( n \times 9 \) products below, note that the digits of the last five products are the same as those of the first five products, except they are in reverse order.

4. The sum of the digits of \( n \times 3 \) or \( 3 \times n \) products is 3, 6, or 9. The sum of the digits of \( n \times 9 \) or \( 9 \times n \) products is always 9.

5. There are repeating patterns in the ones digit of the products (e.g., for \( 4 \times n \) the pattern: 0, 4, 8, 12, 16, 20, 24, 28, 32, 36).

Key for Probe 5.5 (page 5-25)
6. The difference between the products of successive doubles forms an arithmetic progression:

\[
\begin{align*}
2 \times 2 &= 4 & \text{Difference} &= 5 \\
3 \times 3 &= 9 & \text{Difference} &= 7 \\
4 \times 4 &= 16 & \text{Difference} &= 9 \\
5 \times 5 &= 25 & \text{Difference} &= 11 \\
6 \times 6 &= 36 & \text{Difference} &= 13 \\
7 \times 7 &= 49 & \text{Difference} &= 15 \\
8 \times 8 &= 64 & \text{Difference} &= 17 \\
9 \times 9 &= 81 & \text{Difference} &= 19
\end{align*}
\]

7. The products of each row sum to a number divisible by 3 (e.g., the second row: \(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45\) and \(45 \div 3 = 9\)).