BASIC MATHEMATICAL TOOLS: NUMBERS AND NUMERALS

TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 4•1: Number Concepts and Counting Skills

The aims of Unit 4•1 are to help students (a) explicitly understand the various meanings of number, (b) appreciate the task children confront in learning even basic counting and number competencies, and (c) become familiar with developmentally appropriate methods for fostering these basic competencies.

Meanings of Number. To deepen their own number sense and to ensure they can help their own pupils construct a broad concept of numbers, students must become proficient in distinguishing among the cardinal, measurement, ordinal, and nominal meanings. Analyzing examples of number uses (e.g., Part I of Probe 4.1: Meanings of Numbers and Uses of Counting on page 4-4 of the Student Guide) can be useful in achieving this aim and helping students recognize that numbers often have multiple meanings (e.g., the 4 in chapter 4 above has both an ordinal meaning because it can help locate the chapter in this guide) and a nominal meaning (e.g., the 4 can also serve as a chapter label).

Task Confronting Children. Students typically are not cognizant of the complexity of such basic counting skills as orally counting to 100 or enumerating a collection. The cognitive analyses of basic counting and number tasks (Part II of Probe 4.1 and Probe 4.2: An Analysis of Oral-Counting Skills on pages 4-4 to 4-7 of the Student Guide) should help them better understand what young children are attempting to master and provide a powerful framework for analyzing a pupil’s current developmental status and for planning instructional activities that will foster further development. Students may need help distinguishing between the two types of object counting: enumeration and set production. Briefly, the former involves counting a collection to determine an amount (its cardinal value), and the latter is essentially this process in reverse (starting with a cardinal value and counting out a representative collection from a pile of items).

Teaching. Because many people consider learning the counting sequence a rote task, it is important to underscore that a teacher should focus on helping children (a) discover counting patterns and (b) explicitly recognize exceptions to these patterns. Error-detection activities (see Box 4.3 on page 4-12 of the Student Guide) can be one enjoyable way of accomplishing these aims. It is also probably important to emphasize that the learning and practice of various counting and number skills can be made purposeful by using games, everyday situations, and children’s literature. Analyzing examples of each in terms of the specific counting and number skills and concepts involved can be a powerful way of reinforcing the chapter’s content and the value of a purposeful approach. This is the aim of Probe 4.3: An Analysis of the Number Instruction in the Mathematics Their Way (MTW) Curriculum (page 4-15 of the Student Guide).

Unit 4•2: Numeral Literacy

As Parts I and II of Probe 4.4: Understanding Numerical Skills (page 4-17 and 4-18 of the Student Guide) illustrate, there is considerable disagreement and confusion about the teaching and learning of numeral reading and writing skills. For example, the authors of one recent elementary mathematics methods textbook recommended delaying numeral-writing instruction until first grade when children have developed the fine-motor coordination necessary for this skill. There is, however, no theoretical or empirical reason to take this recommendation seriously. Unit 4•2 further underscores that teachers need a powerful theoretical...
framework in order to make informed and effective educational decisions. **Part III of Probe 4.4** (pages 4-18 and 4-19) is intended to help students build such a framework for understanding the teaching of learning of numeral-literacy skills. Figure 4.1 below summarizes what is required for these skills.

**SAMPLE LESSON PLANS**

Instructors who feel that numeral skills are important to cover may choose to have two lessons on Chapter 4 material—one on Unit 4•1 material and one on Unit 4•2 content. Some instructors, particularly those teaching an early childhood mathematics methods course, may wish to spend more than two class periods on this topic.

**Project-Based Approach**

Using SUGGESTED ACTIVITIES on pages 102 to 104 as a menu, have small groups of about four students choose a project. Fairly comprehensive coverage of chapter 4 content can be achieved by having the groups choose different projects and report on them to the class.

**Single-Activity Approach**

**Lesson 1.** Analyzing a primary-level textbook or curriculum can provide a purposeful way of introducing Unit 4•1 content. One way of accomplishing this is to undertake **Probe 4.3: An Analysis of the Number Instruction in the Mathematics Their Way (MTW) Curriculum** (page 4-15 of the Student Guide). To supplement this analysis or to provide a more open-ended worthwhile task, a class can undertake Suggested Activity 11—either as a whole class or in teams—and follow-up with a whole-class discussion.

**Lesson 2.** **Probe 4.5: Analyzing Numeral-Writing Instruction** (page 4-24 of the Student Guide or Suggested Activity 11 can provide a reason for exploring the material in Unit 4•2.

**Multiple-Activities Approach**

**Lesson 1.** For fairly comprehensive and systematic coverage of Unit 4•1 material, an instructor can choose, for instance, the following sequence of activities:

1. **Probe 4.1: Meanings of Numbers and Uses of Counting** (pages 4-4 and 4-5 of the Student Guide) can serve to help students construct an explicit understanding of the meanings of number (Part I) and familiarize them with the names of various counting skills (Part II). Question 2 of Part II can help students better appreciate the following hierarchical development of counting skills: comparing two numbers depends on (at least implicitly) understanding the cardinality rule; accurately applying the cardinality rule depends on accurate enumeration; and accurate enumeration depends on coordinating the automatic skills of generating the correct number-word sequence, pointing one to one at each item (labeling each item in a collection with only one counting tag), and keeping track of which items have already been counted and which haven’t.

2. **Part I of Probe 4.2: An Analysis of Oral-Counting Skill** (pages 4-6 and 4-7 of the Student Guide) can help students explicitly recognize that, unlike our written number system, the English number-word sequence is only highly regular, that exceptions to the counting pattern are a main source of oral-counting difficulties, and that oral-counting instruction should focus on helping children discover patterns (induce counting rules) and recognize exceptions to these patterns. **Part II** can help familiarize students with other oral-

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**Figure 4.1: Summary of the Prerequisites for Numeral-Literacy Skills**

<table>
<thead>
<tr>
<th>MENTAL IMAGE</th>
<th>MOTOR PLAN</th>
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<tbody>
<tr>
<td>Required to recognize or read a numeral</td>
<td>Required to write a numeral</td>
</tr>
<tr>
<td>Parts</td>
<td>MOTOR PLAN</td>
</tr>
<tr>
<td>Part-whole relationships</td>
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</tr>
<tr>
<td>Left-right orientation</td>
<td></td>
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</tbody>
</table>
counting skills, such as automatically stating the number after another.

3. Question 3 of **Probe 4.3: An Analysis of the Number Instruction in the Mathematics Their Way (MTW) Curriculum** (page 4-15) can serve as a basis for discussing what enumeration instruction should focus on in kindergarten.

**Lesson 2.** For comprehensive and systematic coverage of Unit 4•2 material, an instructor could choose to do the following sequence of activities:

1. Without providing feedback afterwards, have students complete **Part I of Probe 4.4: Understanding Numerical Skills** (on page 4-17 of the **Student Guide**). An instructor can then make the point that how teachers think about and respond to a numeral-reading or -writing difficulties depends on their theory of learning and, without an accurate theoretical framework, teachers will probably waste their and their pupils’ time and effort.

2. **Part II** of the probe (pages 4-17 and 4-18) was designed to help students see that traditional view of numeral-reading or -writing difficulties does not make sense. More specifically, if Joey’s writing difficulty is due to perceptual confusion, how does he know that the backward 7 he drew is wrong? How can Joey’s reversal be due to a motor difficulty when writing a backward 7 requires as much motor skill as writing a 7 correctly? Why is copying a numeral not easier than writing a 7?

3. **Part III** (page 4-18) introduces a research-based model for how numeral skills develop. This model suggests that Joey recognizes that his backward 7 is wrong because he has an accurate mental image of the numeral and that his numeral-reversal problem is due to an inaccurate motor plan (a cognitive plan for translating his mental image into motor actions). Copying a numeral is not easier than writing a numeral because both require a motor plan. For example, copying Figure A is not a simple task. Unless you have a preplanned course of action that specifies where to start, what direction to head off in, when to stop, what to do next, and so forth, it is unlikely you will be able to copy the figure correctly. Note that some students confuse a motor plan with a mental image or aspects of a mental image (e.g., part-whole relationship). That is, they seem to assume that the mental image of a numeral specifies how to write the numeral. In fact, young children can have a completely accurate mental image of a numeral and still have no idea about how to write it (a motor plan for translating it into motor actions).

Another potentially confusing aspect of the model is that children do not need a completely accurate mental image to identify and read numerals. For example, to identify a 6, all a child needs to know are its parts and part-whole relationships. This will allow the child to distinguish a 6 from all other numerals, but not a correct 6 from a backward one. Writing a numeral correctly, on the other hand, requires a completely accurate mental image, including left-right orientation (as well as a motor plan).

4. Working through **Parts IV and V** (page 4-19 of the **Student Guide**) in small groups and then discussing them as a class can help students consolidate their understanding of the research-based model.

5. As a class, revisit the vignettes in **Part I**, analyze the difficulties and instructional recommendations in terms of the research-based model.

**SAMPLE HOMEWORK ASSIGNMENT**

**Lesson 1**

Read: Unit 4•1 in chapter 4 of the **Student Guide**.

**Study Group**

- **Questions to Check Understanding**: 1, 2a to 2q, 3, 4, 5, and 6 (pages 104 to 106).

- **Problem**: Clock Chimes (pages 109).

- **Bonus Problem**: Increasing Numbers (page 109).

**Individual Journals:** Writing or Journal Assignment 2 (page 108)

**Lesson 2**

Reading: Unit 4•2 in chapter 4.

**Study Group**

- **Questions to Check Understanding**: 2r to 2t, 11, and 12 (pages 105 to 107).

- **Writing or Journal Assignment**: 9 (page 109).

- **Problem**: Archery Practice (page 109).

- **Bonus**: Writing or Journal Assignment 10 (page 109).
FOR FURTHER EXPLORATION

ADDITIONAL READER INQUIRIES

Probe 4.A (pages 95 to 98)

Analyzing Object-Counting Difficulties and the next probe might be particularly useful for an early childhood mathematics methods course, where a more in-depth exploration of counting and number concepts and skills is desired. Because enumeration requires (a) knowing the number sequence, (b) assigning a single number to an object, and (c) keeping track of which objects have been counted and which need to be counted, it follows that there are three basic kinds of enumeration errors. Probe 4.A was designed to help students recognize these three types of object-counting errors. Note that it may help to actually model or show videotape of enumeration errors and have students try to identify the types of errors in real time (as they happen).

Probe 4.B (pages 99 to 101)

Two Views of Initial Number Development and Mathematics Instruction requires students to compare and contrast the Logical-Prerequisite and Counting Views of number-concept development. It can also involve students in a detailed analysis of counting principles.

QUESTIONS TO CONSIDER

1. Children typically learn to count by twos ("two, four, six, eight . . . ") before they learn the odd-number sequence ("one, three, five, seven . . ."). Why?

2. Young children are sometimes given "dolled-up" enumeration tasks, which entail coloring, cutting, or drawing. For example, a worksheet may outline 8 kick balls and instruct the child to color the kick balls, count them, and circle the appropriate numeral. Would tasks like those described above help or hinder children's enumeration effort? Consider, in particular, the most common type of difficulty kindergartners have enumerating collections.

3. In evaluating children's ability to determine the larger of two numbers, why is it important to randomly present half the comparisons with the larger number last (e.g., which is more, 5 or 6?) and the other half with smaller number last (e.g., which is more 6 or 5?).

4. Silvia, a first grader, had no trouble reading numerals but did have difficulty writing 2, 7, 6, and 9. At first, she had written in a peculiar fashion ə , 7 as ə , 6 as ə , 9 as ə . Interestingly, she had similar difficulties with K, N, R, W, X, Y and Z. What might be the source of her difficulty?

5. (a) What should a teacher do if a child has an unorthodox motor plan that yields a legible numeral? For example, some children begin their nines at the bottom: ə . Should such children be encouraged to adopt the standard motor plan? (b) There are two ways to make a four: 4 and 4. Is it preferable to teach one way of writing four over the other?

6. Leonard, a third grader, already knew how to write most of his letters but had difficulty writing all the numerals but 1 and 8. Consider how a teacher could build on his existing motor plans for the letters to help Leonard construct motor plans for the numerals. What letters provide partial motor plans for numerals and how could such plans be supplemented for numeral writing?

7. Constructing a mental image of a numeral entails finding the commonalities among diverse examples. For example, the numerals 9, 9, 9, q, and 9 all share the same features: a circle-like form attached to the upper-left hand part of a line. (a) Discovering the commonalities among various examples of nine is what type of reasoning? (b) Using a mental image (the abstracted commonalities) to recognize new instances (e.g., 9, 9, or 9) entails what kind of reasoning?

8. (a) Consider Sally’s nose analogy for writing sixes (Figure 4.2 on page 4-16 of the Student Guide). Will it help prevent reversals? (b) What analogies for writing numerals can you think of?

9. Asked to draw an eight, 4-year-old Arianne asked, “How do (text continued on page 102)
Once children learn some number words, they set about the task of using these number words to count (enumerate) collections—to determine the number of items in collections. The aim of this probe is to help you construct an explicit understanding of children’s difficulties with enumeration so that you are in a better position to guide their learning.

Types of Enumeration Errors*

Three basic types of enumeration errors (sequence errors, coordination errors, and keeping-track errors) are described below and illustrated in Figure 4.2 on the next page.

1. **Sequence errors** occur when a child does not use the correct counting or number-word sequence (e.g., tagging three blocks: “One, two, ten”).

2. **Coordination errors** involve a break down between oral counting and pointing (e.g., pointing to an item but not tagging it with a number word or pointing to an item while saying more than one number word). Children just learning to enumerate sets and those with learning difficulties, in particular, may have difficulty starting or stopping the oral-counting and pointing processes at the same time (Gelman & Gallistel, 1978). As a result, a child may point to the first item and say nothing but, thereafter, honor the one-to-one principle. Likewise, a child may get to the last item of, say, a five-object set and let several tags slip out before stopping the oral count (e.g., “five, six, seven”). Children are especially prone to coordination errors when they rush their counting.

3. **Keeping-track errors** involve failing to keep track of which objects have been counted and which need to be numbered, and, thus, skip an object or count it more than once. Such errors are the most common type of enumeration error among children just starting school. Apparently, some preschoolers fail to devise effective keeping-track strategies, such as creating a separate pile for counted items (Fuson, 1988). Keeping-track errors are also more likely if a child rushes.

Multiple errors can occur when several types of errors are made or when one type of error is repeated. Extreme forms of such errors are described below and illustrated on the next page.

- **Skims** involve no effort at one-to-one counting (coordinating oral counting and pointing) or keeping track. Initially, children may simply pass a finger over a collection while spewing out numbers (Fuson & Hall, 1983). Typically, though, children construct an understanding of the one-to-one principle relatively early—well before they begin kindergarten—and make some effort to engage in one-to-one counting. This type of multiple error, then, should be relatively rare among school-age children.

- **Flurries** can involve several types of errors (e.g., a keeping-track error and a coordination error) or multiple instances of one type of error (e.g., several errors in keeping track) or combinations of error types.

Kindergartners who regularly exhibit skims or flurries, particularly with collections smaller than 10, need special attention.

* It is important for a teacher to understand that children have a variety of methods for counting objects and the reasons for these variations. Far less important is knowing the names of each type of error or subtype of error.
Probe 4.A continued
Identifying Types of Enumeration Errors

By observing children’s enumeration efforts and identifying the specific type of error they make, a teacher is in a better position to help them overcome object-counting difficulties. With the aid of the figure on the previous page, identify whether the enumeration error in each sample below is (a) a sequence error, (b) a coordination error, (c) a keeping-track error, (d) a skim, or (e) a flurry. For the last type, specify what types of errors were made.
The Case of Vince: Coming Up Short

Vince, a kindergarten child, was asked to count six poker chips haphazardly arranged in front of him. He did so silently and answered, "There's five."

1. List the three component skills that make up the ability to enumerate. (a) For each component, indicate whether or not a difficulty with that subskill could have produced the enumeration error. (b) Illustrate how the error might have occurred.

2. What does the case of Vince illustrate about evaluating children's mathematical knowledge?

The Case of Billy Bob the Bad Counter

Billy Bob came to kindergarten with little informal mathematical experience. As a result, he had difficulty with basic skills that can be taken for granted in other children. More specifically, he labored to produce the number-word sequence. Given a collection of 5 blue chips and a collection of 4 white chips, he enumerated the blue chips ("1, ah 2, ah 3, ah 4," and the white chips ("1, ah 2, ah 3"). Asked which has more, Billy Bob stared blankly at his teacher and said, "I'm not sure—it could be the blues or it could be the whites—it's kind of close. The blues have 1, 2, 3, 4 and the whites 1, 2, 3."

1. What can be said about Billy Bob's understanding of (a) the cardinality principle, (b) the number-order principle, and (c) the order-irrelevance principle?

2. What enumeration error may Billy Bob have committed in counting the collection of blue chips?

3. Billy Bob's teacher, observing his labored and inaccurate attempts to enumerate collections, recalled that her LD text said something about inattentiveness caused by Attention-Deficient Syndrome (ADS). "That's it," she exclaimed with glee, "I'll give attention-building exercises." Unfortunately, Billy Bob did not prosper under the attention-building regime. Instead of concluding Billy Bob was suffering from ADS and inattentiveness, how else might his teacher interpret his "attention" difficulties while enumerating?

The Case of Hamish the Inconsistent Counter

Hamish, a 5-year-old kindergartner, could orally count up to 29 without any difficulty. At the beginning of the school year, his mathematics instruction focused on counting both orderly and haphazard collections of up to six items. Hamish, an extremely accurate counter, did very well. In December, counting tasks began including orderly and haphazard collections of 6 to 20 items. Suddenly, Hamish became inconsistent—sometimes accurate, sometimes inaccurate. Often his counts were off by two or three. Account for Hamish's sudden inconsistency? What was the main source of his difficulty and what type of counting errors probably contributed to it?

Questions for Reflection

1. Taking into account that understanding cannot be imposed, how would you help a kindergartner who regularly exhibited skims or flurries? How might playing a board game with peers help?

2. Brett tends to make sequence errors such as..."four, five, nine, eight, ten." Casey tends to make coordination errors, often tagging an item with several numbers. Mazid frequently makes a keeping-track error, not counting an item or counting an item twice. How would your efforts to help these three children differ?
According to the Logical-Prerequisites View espoused by Piaget (1965) and others, children must first develop the logical concepts (prerequisite concepts) of classifying and ordering in order to understand number and counting. This view has had a significant impact on primary-level textbooks. Part I of this probe asks you to analyze workbook pages based on this view. Part II asks you to analyze a task in terms of the counting principles proposed by the Counting View as central to the development of a number concept. Part III asks you to reflect on a key piece of evidence supporting the logical-prerequisite view.

Part I: Analysis of Curriculum Materials

Shown on the next page are four workbook pages that involve tasks commonly found in kindergarten and first-grade mathematics textbooks.

1. What skill is addressed by (a) workbook page 1, (b) workbook page 49, and (c) workbook page 57?


3. (a) The intent of workbook page 68 is to have children use a matching strategy to determine the equivalence or inequivalence of the collections and to use number labels (numerals) correctly. Is it likely that matching would be children’s strategy of choice in comparing smaller collections of 2 of 4 items? Is the same true for collections larger than 4? (b) What strategies other than matching might some children use? (c) Psychologically, does it make sense to introduce equivalence in terms of matching rather than counting?

Part II: An Analysis of a Child’s Counting Ability

Jake was playing cowboys and Indians with his brother Paul. Jake counted a row of toy Indians from left to right and concluded that there were "five." He counted the toy cowboys and concluded that there were "six." He turned to Paul and said, "I've got more Indians." Paul asked, "What if you count them the other way?" Jake responded, "I don't know; let me count them." He then proceeded to count the toy Indians from right to left and exclaimed, "Five again!"

(Part II continued on page 101.)
The first six chapters of the kindergarten textbook of \textit{THE MATH BOOK FROM HELL} do not mention counting or numbers. The three workbook pages (pages 1, 49, and 57) depicted below accompany three of these chapters. The fourth workbook page depicted (page 68) accompanies chapter 7: Numbers 1 to 5.

\textbf{LIKE THINGS}

1. Circle all the blocks.

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\end{itemize}

2. Circle all the boats.

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\end{itemize}

3. Circle all the flowers.

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\textbf{PUTTING THINGS IN ORDER}

Instructions: Cut out the pictures of the children below. Then paste them on the picture from smallest to biggest.

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\end{itemize}

\textbf{SAME NUMBER?}

Instructions: For each box, draw lines to see if there are the same number of balls as children. Circle whether there are the same number or not the same.

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\end{itemize}

\textbf{SAME NUMBER?}

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\end{itemize}
Probe 4.B continued

Circle the number of any counting principles Jake demonstrated. Briefly justify why you think a principle was or was not involved. If the evidence did not examine a principle, indicate with N/A (not applicable).

1. Abstraction Principle (even diverse items can be treated as a collection for counting purposes)
2. Stable-Order Principle (the same sequence of count words must be used on every count)
3. One-to-One Principle (one and only one unique count word can be assigned to each item in a collection)
4. Cardinality Principle (the last count word has special significance because it represents the total number of items counted)
5. Number-Order Principle (the later a number word in the counting sequence, the larger the quantity it represents)
6. Identity-Conservation Principle (as long as nothing is added or subtracted, the number in a collection remains the same despite changes in the appearance)
7. Order-Irrelevance Principle (as long as the one-to-one principle is observed, the cardinal value of a collection stays the same despite the order in which items of the collection are enumerated)

Part III: The Case of Peter: An Evaluation of Crucial Evidence

Peter, a preschooler, put seven blue poker chips in a row. A tester placed seven white poker chips in one-to-one correspondence with the blue chips. Then, while Peter watched, the tester put out an eighth white chip and shortened the row of white chips:

Encouraged to count to see if the two rows had the same number or whether one had more. Peter responded, "My row has [counts the blue chips] one, two, three, four, five, six, seven. Your row has [counts the white chips] one, two, three, four, five, six, seven, eight. See your row only has eight—my row has more!" Why didn’t counting help Peter?

1. How would proponents of the Logical-Prerequisites View (see page 4-9 of the Student Guide) explain Peter’s surprising conclusion?

2. How might Peter’s surprising behavior be explained in terms of the counting principles outlined above and the Counting View described on pages 4-9 and 4-10 of the Student Guide? That is, what understanding about counting and numbers had Peter, perhaps, not yet constructed?

Questions for Reflection

1. The outcome of the enumeration process names a collection. What counting principle specifies how this happens and with which meaning of number is this principle associated?

2. (a) The Abstraction Principle would seem to be linked with which meaning of number? (b) The Order-Irrelevance Principle clearly applies only to which meaning of number? (c) The Number-Order Principle involves which meaning of number?
you make it?” Her father suggested a snowman analogy: Draw a circle to make the snowman’s body. Now draw a smaller circle for the snowman’s head and put it right on top of the big circle. After completing these instructions, Arianna asked, “Should I make eyes?” Her father instructed, “Not for an eight. An eight is a snowman without eyes.” What are the advantages and disadvantages of the snowman analogy and do the former outweigh the latter?

**SUGGESTED ACTIVITIES**

1. (a) Create a bulletin-board display or a picture book that illustrates everyday examples of the four basic meanings of number: cardinal, measurement, ordinal, and nominal meanings. Indicate which examples illustrate more than one meaning. (b) Create an activity or performance-assessment item in which primary-level or intermediate-level children would be involved in distinguishing among the four basic meanings.

2. Individually test a small group of kindergartners on their forward oral-counting sequence to forty. Ideally, include children of high, average, and low ability. (a) To ensure interest, devise an oral-counting test that is purposeful (e.g., counting tallies to determine a final score in a game). (Assessing oral-counting in the context of counting real objects for a real purpose can better gauge true success [Fuson, 1988]). (b) For each child, indicate the numbers of counting words given in the correct order. For example, for a count of “1, 2, 3 . . . 29, 37, 38, 39, 40,” the score would be 33; for “1, 2, 3 . . . 29, thirty-ten,” 29; for “1, 2, 3 . . . 14, 16, 17, 18, 19,” 18; for “1, 2, 3 . . . 7, 9, 8,” 7; for “1, 2, 3, 8, 9, 10,” 6; for “1, 2, 3, 11, 8, 9, 10,” 6; for “1, 2, 3, 12, 3,” 3. For each child, indicate what type of oral-counting error was first made: error in the rote sequence to 12, error with a teen exception (13 or 15), error with other teens, error with a decade term (20, 30, or 40), error with the repeating cycle (e.g., not realizing that 21, 22, 23 . . . follows 20), other error with the 20’s, or other error with the 30’s. (c) Compile your data with others in your group to create an object-counting profile (e.g., what enumeration skill can a teacher expect of high-, average-, and low-ability kindergartners at midyear). Compile the error data to identify the most common types of enumeration errors made by kindergartners. (d) From your group’s data, draw up a list of teaching suggestions for each ability level or individual child. (e) Prepare a report, including graphs of your data, and present your results and teaching suggestions to your class.

3. Individually test a small group of kindergartners on their enumeration skills for haphazard collections of 2 to 5 items (small sets) and haphazard collections of 6 to 10 items. Ideally, include children of high, average, and low ability. (a) To ensure interest, devise an object-counting task that is purposeful (e.g., playing a game that involves counting the number of dots on a card in order to determine the number of spaces a race car can move). Note that using fixed items may make the task more difficult for children than using moveable items. (b) For each child, indicate the ratio of correct counts to total number of trials for each size level. Note that it is essential to examine children’s enumeration processes in order to distinguish between a true success and a false success. For example, in counting a collection of five items, a child might skip one item and count another twice and indicate that there are “five.” However, two wrongs do not make a right and, this effort should be scored as incorrect. For each child, also indicate the types of enumeration errors made. See Probe 4.A on pages 95 to 98 of the guide for a description and examples of the five basic types of enumeration errors: sequence error, coordination error, keeping-track error, a skim, and a flurry. (c) Compile your data with others in your group to create an object-counting profile (e.g., what enumeration skill can a teacher expect of high-, average-, and low-ability kindergartners at midyear). Compile the error data to identify the most common types of enumeration errors made by kindergartners. (d) From your group’s data, draw up a list of teaching recommendations for each ability level or individual child. (e) Prepare a report, including graphs of your data, and present your results and teaching recommendations to your class.

4. Use a diagnostic test such as the TEMA-2 (Ginsburg & Baroody, 1990) to assess the counting ability and number sense of three kindergartners and three first-graders. Ideally, you should include a high-, average-, and
low-ability child at each grade level. For the TEMA-2, items 3, 6, 8, 9, 10, 11, 17, 18, 20, 21, 24, 27, 28, 30, 32, 33, 34, 36, 42, and 61 assess the following counting competences: oral (forward) counting, enumeration, cardinal rule, production of fingers or objects, number after, counting backwards, and skip counting (by tens and fours). Items 1, 4, 7, 16, 23, 31, and 51 assess the following aspects of number sense: perception of more, number identity, next-number comparisons, and distant-number comparisons. What are the instructional implications of your results?

5. (a) Devise a kindergarten screening test that gauges the following skills: forward counting, enumeration, cardinality, set production, and number order (both distant-number and next-number comparisons to 10). The test should be entertaining for children and consists of no more than four tasks. (A task may entail more than one skill.) The test should be brief—take no more than 10 to 20 minutes. (b) Discuss your test with your group and class and revise it as needed. (c) Administer the test to a handful of kindergartners. Record and summarize your results. (d) Report your findings and conclusions to your class.

6. (a) Play an error-detection game (see, e.g., Box 4.3 on page 4-12 of the Student Guide) with a group of peers to practice counting in base four. Start with two-digit sequences (e.g., "In base four, can I count 'two-one, two-two, two-three, two-four'?"). Then proceed to three- and four-digit sequences. Take turns as the instructor. Try to use errors that will encourage the participants to think about the structure of the base-four sequence (e.g., "three-one, three-two, three-three, four-zero"). Instructors should ensure that those least experienced with base systems are the ones most often asked to respond. (b) Play an error-detection game that involves oral-counting errors and correct counts up to 100 with a group of kindergartners. Describe your examples (including correct ones) and how the children responded to each. Did all, most, some, or only a few appear to catch an error? Evaluate the effectiveness of your lesson. Did the game help the weaker counters? Specify the evidence supporting your conclusion.

7. (a) Compile a list of children's books that involve oral-counting competencies (e.g., forward, backward, and skip counting) or object-counting competencies (e.g., enumeration, cardinal rule), or numeral skills (e.g., identifying or reading numerals). For each book, explain what basic skills or concepts are involved and how. That is, explain how a parent or teacher could use the book to demonstrate a skill or have a child practice it.

8. Observe several kindergartners as they complete the task below. Note how many children use direct perception (compare the length of the two rows), how many count, and how many use matching. Indicate the number in each category who were successful. Pool your data with others in your group or class, perhaps summarizing your results in a graph. Discuss which strategy children typically used and which they used most successfully. What are the instructional implications of your results?

Are there the same number of balls as bats?

9. How problems could be used as a vehicle for raising questions and exploring content was discussed or illustrated in chapters 1 to 3 of the Student Guide. Find or devise a problem that would be useful for introducing one or more concepts discussed in this chapter.

10. Create a rap for teaching children motor plans for the numerals 1 to 9.
11. Choose a textbook or a primary-level curriculum. (a) Analyze it in terms of the instructional recommendation described in chapter 4 of the *Student Guide*. Include in your analysis the counting and number concepts and skills described in Unit 4•1 and the numeral-reading and -writing skills, described in Unit 4•2. For example, does the instruction help children construct an understanding of all four meanings of number? Does it provide adequate instruction and practice of the next-number comparison skill? Does numeral-writing instruction focus on helping children construct a motor plan? (b) Does the program most closely resemble the skills, conceptual, problem-solving, or investigative approach? (e) Describe and justify your recommendations for changing or supplementing the program.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. (a) Mindy, who was actually 40-years-old, was celebrating her 35th birthday. Her birthday cake required 4 cups of frosting and was adorned with 3 large candles (to represent decades) and 5 small candles (to represent years). Identify which meanings of number are exemplified by 40, 35, 4, 3, and 5 in the preceding sentence. Briefly justify your answers. (b) The title on page 4-1 of the *Student Guide* includes chapter 4. Evaluate this use of the number 4 in light of the four meanings of numbers discussed in the chapter. Briefly justify why each meaning is or is not applicable. Recall that the guide starts with chapter 0.

2. Answer the following modified true-false questions. Circle the letter of any statement that, according to this guide, is true. Change the underlined portion of any false statement to make it true.

   a. Learning to count from 14 to 29 entails [rote memorization].

   b. Among typical kindergartners, the most common error in counting up to 19 is "eleven and twelve."

   c. A common error at the end of the twenties series is [restarting with one] (i.e., ... twenty-eight, twenty-nine, one ...].

   d. Children learn to cite the number after another at the same time they learn that portion of the number sequence.

   e. Children learn to count forward before they learn to count backward.

   f. Enumeration involves making a one-to-one correspondence between each counting-sequence number and each [finger point].

   g. Among children entering kindergarten, the most common enumeration error in counting collections of up to 10 items are [counting-sequence] errors. Therefore, kindergarten teachers need to focus on teaching the correct terms 1 to 10 in the proper order.

   h. Recognition of number patterns such as \( \circ \circ \circ \circ \) ("Four") and \( \circ \circ \circ \circ \) ("Five") should be encouraged.

   i. Finger counting and finger patterns should be [discouraged].

   j. Numerical relationships involve one or more collections.

   k. Matching is children’s natural (informal) and first way of accurately establishing the equivalence of collections with five or more items. Therefore, primary instruction should build upon this skill when teaching about equivalence.

   l. Children discover quantity-comparison relationships such as the same number-name and ordering-numbers principles from [one-to one matching] experiences.

   m. Kindergarten teachers can safely assume that nearly all children know next-number number comparisons to 10.

   n. For numbers between 5 and 10, children can make next-number comparisons (determine the larger of two neighboring numbers such as 5 and 6) before they can accurately make distant-number comparisons (e.g., determine whether 5 or 9 is more).
o. In teaching preschoolers next-number number comparisons, it is helpful to relate this skill explicitly to their number-after knowledge.

p. Requiring a child to orally count to 10 would be a valid kindergarten screening task for evaluating a number concept (an understanding of number).

q. Oral- and object-counting skills typically develop before numeration skills (reading and writing numerals).

r. A child who can recognize 7 as seven and indicates that \( \nabla \) is backwards but who cannot write the numeral properly probably lacks part-whole knowledge: How the parts fit together.

s. On a numeral-reading task, young children are most likely to confuse the numeral 2 with \( \underline{1} \), because they are so closely linked in the counting sequence.

t. To minimize confusion, it is best to teach children how to read and write the numerals 6 and 9 separately—preferably at least six months apart.

3. Circle the letter of each of the following statements that, according to this guide, is true.

a. Correct oral counting to 10 implies object counting ability with collections up to 10.

b. Learning to count by tens (the decades sequence) probably entails both rote memorization and pattern recognition (rule-governed learning).

c. Some children stop at 100 when counting by ones, because they do not realize that the hundreds are simply a repetition of the count to 99 with the prefix "one-hundred" added.

d. Instruction on oral counting should emphasize drill and practice.

e. To make it interesting, oral counting should be practiced in the context of object counting.

4. For a kindergarten screening test, children are asked to orally count to 30. According to the

Student Guide, there are three different types of oral-counting errors (stumbling blocks) for many 4- and 5-year olds. Note the five numbers typical kindergartners are most likely to miss. Briefly justify your choices in terms of the three different types of counting errors such children are prone to make.

5. Entering kindergartners were administered a screening test that included the following test item: Which is more, 8 or 9? 7 or 6? 9 or 10? and 8 or 9? A child simply guessed, got only two correct answers, and failed the test. Circle below more basic skills that should be checked to ensure that the child has the bases for making next-number comparisons with numbers from 6 to 10.

a. Oral count sequence to 10

b. Decade sequence to 100 (10, 20, 30 . . . )

c. Distant-number comparisons (e.g., Which is more 2 or 9?; 10 or 3?)

d. Small-number next-number comparisons (e.g., Which is more 2 or 3?; 4 or 3?)

e. Number after 1 to 9 (e.g., What number comes after seven?)

f. Number before 1 to 9 (e.g., What number comes before seven?)

g. Counting backwards from 10 to 1

6. Ms. Prim, your principal, was shocked that your kindergartners were playing games during class time. You pointed out that there was considerable academic value in playing games because children can learn and practice mathematical concepts and skills. "What's the value of this," demanded Ms. Prim, pointing to the game Star Collector (see Figure 4.3 on the next page). After each counting principle or skill listed below, indicate the game step below (1, 2, 3, 4, 5, or 6) that could be used to demonstrate or practice it.

a. Cardinality Principle

b. Enumeration

c. Set Production

d. Number-ordering principle
(1) Rolling a dot die and determining how many dots came up.
(2) Mentally note the number counted.
(3) Moving a marker the number of spaces specified by the die roll.
(4) Upon landing on a square with stars, determine how many stars there are.
(5) Collects that number of "star markers."
(6) In some situations, chooses which of two paths to take to maximize his/her winnings.

Figure 4.3: Star-Collector Board

7. The steps in the game Soccer (Activity File 4.5 on page 4-14 of the Student Guide) are listed below. (a) In the blank by each step, indicate whether it involves enumeration (note with an E), the cardinality principle (C), number-pattern recognition (N), and/or set production (P). Note that more than one principle or skill may be applicable. (b) For each step, justify your answer(s).

i. On their turns, the players draw a dot card and determine how many dots are on it.

ii. The drawn number determines how many spaces they can move.

8. Finger patterns (e.g., simultaneously raising five fingers to show "five") is a special case of which of the following (circle the letter of the one best answer): (a) oral counting, (b) enumeration, (c) set production, (d) ordering numbers.

9. Aspects of the Grade K text of a popular elementary mathematics textbook series are summarized below. Indicate whether or not each aspect is consistent with the developmentally based recommendations of the Student Guide. Briefly justify your answers.

a. Starting with the first lesson of chapter 1, each lesson begins with a daily-count activity: activities that involve oral and/or object counting.

b. The oral-counting activities are introduced before object-counting activities.

c. Counting forward to a number is introduced before counting backward from a number.

d. Written numbers are not introduced until chapters 6 to 8.

e. Comparing groups to determine same or different is introduced in terms of one-to-one correspondence (e.g., drawing lines between honey bees and hives).

f. A teacher is instructed to model how to write a number and to ask students to describe the motions used.

10. The following counting skills normally develop in what order: (a) counting backward, (b) number-after a term, (c) number-before a term, (d) forward-counting sequence?

11. Steven, an 11-year-old moderately retarded child, could already recognize and read the numerals 1 to 10. However, except for 1, 2, and 4, he was unable to write the single-digit numerals. One problem that plagued Steven was reversals. Asked to draw a seven he drew and commented "Oh, that is a 't,' this isn't right." For "three," he drew and said, "This is a 's.' How do you make a three?" Assume that Steven could identify backward numerals like as incorrect. Which of the following is/are probably true of Steven's numeral-writing difficulty? Circle any correct statement.

a. He does not know the parts that make up a 7 or 3—a key aspect of their mental images.

b. He does not know the part-whole relationship of 7 or 3—a key aspect of their mental images.

7. The steps in the game Soccer (Activity File 4.5 on page 4-14 of the Student Guide) are listed below. (a) In the blank by each step, indicate whether it involves enumeration (note with an E), the cardinality principle (C), number-pattern recognition (N), and/or set production (P). Note that more than one principle or skill may be applicable. (b) For each step, justify your answer(s).

i. On their turns, the players draw a dot card and determine how many dots are on it.

ii. The drawn number determines how many spaces they can move.

8. Finger patterns (e.g., simultaneously raising five fingers to show "five") is a special case of which of the following (circle the letter of the one best answer): (a) oral counting, (b) enumeration, (c) set production, (d) ordering numbers.

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a. He does not know the parts that make up a 7 or 3—a key aspect of their mental images.

b. He does not know the part-whole relationship of 7 or 3—a key aspect of their mental images.
c. He has a completely accurate mental image of 3s and 7s.

d. He probably has a perceptual-motor deficit, such as fine-motor, visual-motor coordination and/or visual discrimination difficulties.

e. He lacks a motor plan for 3 and 7.

f. He is neurologically impaired and cannot correctly orient 3 and 7 in his visual field.

g. He lacks intrinsic motivation to learn.

12. According to the guidelines suggested by the Student Guide, which of the following might help redress or remedy Steven’s underlying problem? Circle any correct recommendation.

a. Repeatedly have him trace his finger over felt cut-outs of a 3 and a 7.

b. Make a clay model of the numerals 3 and 7, while the teacher points out the parts e.g., a seven is made up of a straight line and a diagonal.

c. Perceptual-motor training designed to improve, for example, fine-motor control, visual-motor coordination, and visual discrimination.

d. Have Steven follow with a crayon as the teacher traces a 5 or 7 with a pencil and describes how she is writing the numeral.

e. Ask Steven to write numerals as he describes the step-by-step procedure for making the numeral.

13. Shown a number line that starts with 1, Jerry paused and then pointed to the numeral 5 when asked: "Which was five?" Miss Peach concluded that Jerry could identify 5—had at least a partially correct mental image of five. Evaluate Miss Peach’s conclusion.

14. Shown the numerals 0 to 9 written correctly, backwards, and incorrectly in random order, Erik identified all numerals correctly except 6 and 9. He identified 6 as "six" about half the time and as "nine" the rest of the time. Except for 6 and 9, he identified all backward numerals, such as \( \_
\), as incorrect. He considered incorrect numerals such as 10 (an incorrect six) and 9 (an incorrect nine) as silly, as not even numbers. According to the Student Guide, which of the following is/are probably true? Circle any correct statement.

a. He knows the parts that make up a 6 or 9—a key aspect of their mental image.

b. He knows the part-whole relationships of 7.

c. He knows the part-whole relationships of a 6.

d. He knows the left-right orientation of 7.

e. He knows the left-right orientation of 6.

f. He has a completely correct mental image of 7.

g. He has a completely correct mental image of 6.

h. It follows from the evidence that he has an incomplete motor plan for 6.

i. He has a perceptual-motor deficit, such figure-ground discrimination difficulties.

15. Circle the letter of any statement that is correct according to the Student Guide.

a. Shown a 6, Nora said: "That's a six." Shown a 9, she said, "That's a nine." Nora has a completely correct mental image of 6 and 9, including left-right orientation.

b. Bert was asked to write a "seven." He wrote \( \_
\), \( \_\), \( \_\) each time indicating it was wrong. His teacher wrote 7 on the board. That's it—that's what I'm trying to write!" Bert exclaimed. Bert has a completely accurate mental image, including left-right orientation.

c. Bert (see part b above) probably lacks knowledge of a seven's part-whole relationships.

d. Bert (see part b above) probably lacks a motor plan.

e. Having Bert (see part b above) copy sevens with green horizontal line and red diagonal (as done in Mathematics Their Way) would probably help him.
WRITING OR JOURNAL ASSIGNMENTS

1. What meanings of numbers have you used today? Illustrate and justify.

2. Mr. Baker introduced his lesson on statistics by noting that two types of data can be collected—numerical and categorical data. For example, responders on a survey can answer some questions (e.g., What is your age?) with a number and some questions (e.g., What is your favorite dessert?) with a word (the name of a category). Alexi excitedly added, “Like 1 for boy or 2 for girl would be numerical data.” (a) What meaning of number did Alexi imply with his example? (b) Is it consistent with the meaning(s) of number Mr. Baker implied for numerical data? Why or why not?

3. (a) What counting skills have you used today? (b) Did you have to exert much mental effort to execute each skill or not? (c) Were you even consciously aware that you were using a skill while executing it? (d) What have you learned about your own counting skills as a result of reading chapter 4?

4. Described below are some counting activities from a popular elementary mathematics textbook. Evaluate each activity and answer the short-answer questions about each.

   a. Children are instructed to count to six as the teacher points to (six) objects on the shelves: “one, two, three, four, five, six.” They are next instructed to point to the objects themselves and count again: “one, two, three, four, five, six.” Then children are asked to count the objects once more as a classmate points. (i) This activity involves practicing what counting skills? According to the Student Guide, would this activity be profitable for a typical kindergartner? Briefly justify your answers. (ii) Does this activity involve the cardinality principle? Why or why not? (iii) Evaluate whether or not children would see this activity as purposeful.

   b. Children are asked to count a collection of blocks as the teacher points to and pushes away each block. (i) What enumeration skill does the italicized portion above demonstrate? (ii) According to research, is this a skill for which many typical kindergarten children need instruction?

5. Examine the worksheet illustrated below and answer the following questions. (a) To answer the "Now how many?" questions in Questions 1 to 3, would a child necessarily have to know the number after a term up to 20? Briefly explain why or why not. (b) Evaluate whether this worksheet introduces one more in a meaningful fashion. Why or why not? (c) Does it introduce one more in a purposeful fashion? Why or why not? (d) Question 4 is designed to practice what counting skill discussed in the Student Guide? (e) Does it make sense to introduce Question 4 after Questions 1 to 3? Briefly justify why or why not.

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**Worksheet: One More**

4. Write the next number. 9 12 19 15 13 18

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6. In a popular Grade K textbook, children are instructed to determine more or less by placing a cube on each item of one collection, doing the same for the second collection, and then lining up the two rows of blocks to see which is longer. The example below is similar to those that appear in the worksheets:

Place cubes on the objects, compare the groups, and circle the box having more.
(a) What is the aim of the activity above? (b) Is the approach above consistent with that recommended by the Student Guide for initially teaching more and less? Briefly explain why or why not?

7. Svein, a first grader, initially wrote a 5 backward. When asked by the teacher if he had written a two, he said, "No, it's a five." The teacher responded, "It looks like a two to me." The student then changed the numeral so that it had the correct orientation. What can you conclude about Svein's difficulty? Can a difficulty with his mental image be dismissed? Why or why not? What can you conclude say his motor plan for 5?

8. While student teaching, Raquel observed a second-grade class complete a worksheet on basic number combinations. Reto, who was one of the brightest children in the class, gave her his paper to correct. Pointing out several backwards 7s, the boy told the student teacher that he sometimes wrote sevens backwards. He wanted to make sure that the student teacher would not mark his answers wrong. Interestingly, some of the 7's had been written correctly. The student teacher could not understand why Reto could not write the number correctly if he knew he was writing it incorrectly. She asked him to go back to his seat and correct all the 7's he had written backwards. He told the student teacher that he did not have to because his teacher said it was okay for him to do that. The student teacher didn't argue. (a) What was the basis of Reto's problem and how could a teacher have helped the boy? (b) How does your diagnosis explain why he sometimes wrote 7s correctly and other times backward?

9. Leo, a third grader, did not have any difficulty reading the numerals 0 to 9. However, he regularly switched between writing the numerals 2 to 7 and 9 correctly and writing them backwards. (The numerals 1 and 8 cannot be reversed.) He effectively had two ways of writing most of the numerals. Moreover, he did not distinguish between correct and backward numerals. For instance, he identified both 7 or \( \sqrt{ } \) as seven (did not know which was correct). (a) From the evidence presented above, can you conclude that Leo knew the parts comprising the single-digit numerals 2 to 7 and 9? Briefly justify. (b) Can you conclude that Leo knew their part-whole relationships except for orientation? That is, except for left-right orientation, did he have a complete mental image of these numerals? Briefly justify. (c) Can you conclude that Leo had a completely accurate mental image of the numerals 2 to 7 and 9, including left-right orientation? Briefly justify. (d) Leo could identify 6 and 9 correctly. That is, he consistently read 6 as six and 9 as nine. How could Leo distinguish between these two numerals? (e) Could he have an accurate motor plan for 2 to 7 and 9? Why or why not?

10. Pico, a first grader, wrote "three" in the following manner: \( \sqrt{ } \). "Uh-oh," he commented, "it's laying down. That's not right" (a) Might his difficulty be due to a faulty mental image? Briefly explain why or why not? (b) Might the boy's difficulty be due to a faulty motor plan? Briefly justify why or why not. (c) Indicate how you would test Pico to show that his difficulty either was or was not due to a faulty mental image. How would you test to determine whether his difficulty was or was not due to a faulty motor plan? Justify your answers.

PROBLEMS

■ Archery Practice (● 4-8)

Robin shot six arrows at a target with rings worth 1, 3, 5, 7, and 9 points. All his arrows hit the target. Which of the following could be Robin's total score: 5, 19, 25, 36, 47, or 56?

■ Clock Chimes (● 4-8)

A clock chimes once at 1 A.M. and 1 P.M., twice at 2 A.M. and 2 P.M., three times at 3 A.M. and 3 P.M., and so forth. During the course of the day, how many times does a clock chime? Assume that a day begins at 12:01 A.M. and ends at 12 midnight.

■ Increasing Numbers (● 6-8)

The numbers 1234 and 1357 can be called "increasing numbers," because any of their digits are bigger than the digits to the left. How many increasing numbers are there between 2000 and 7000?
ANSWER KEY For Student Guide

Key for Probe 4.1 (pages 4-4 and 4-5)

Part I

1. (a) measurement; (b) nominal; (c) cardinal; (d) ordinal

2. The catalog number serves to identify or label a stamp and, hence, has a nominal meaning. Because the number can serve to locate the description of a stamp in the catalog (e.g., the description of the Lindbergh stamp follows that of 1709 and precedes that of 1711), it also has an ordinal meaning. (Many everyday uses of numbers can have multiple meanings.)

Part II

1. (a) ii; (b) iii; (c) i; (d) iv; (e) ii; (f) iii (the child is counting out spaces).

2. (a) According to the ordering-numbers principle, twelve is larger than eleven becomes it later in the counting sequence. (b) The cardinality principle specifies that the last counting tag represents the whole collection. (c) Enumeration requires knowing the number sequence, one-to-one tagging, and keeping-track strategies.

Key for Probe 4.2 (pages 4-6 and 4-7)

Part I

6. (a) Katie, at least implicitly, recognized that nines signal the end of a series, that decades serve as the first term for a new series, and that each new series consists of combining a decade term and the one-to-nine sequence. She also apparently knew the teens are different from later series (the decade designation is last, not first), the teen-rule exceptions (thirteen and fifteen) and the decade exceptions (thirty and fifty). (b) Katie had not yet mastered citing the decade-after another decade (e.g., the decade after seventy is eighty). This shortcut for counting by tens would eliminate her need to count by ones to determine a decade (e.g., count “1, 2, 3, 4, 5, 6” to determine sixty) and would greatly facilitate counting to one hundred. (c) The most common errors include the teen exceptions (thirteen and fifteen) and the decade exceptions (thirty and fifty).

Key for Probe 4.3 (page 4-15)

1. This instruction is based on the Logical-Pre-requisite View—the assumption that children must develop a classification concept before working with numbers directly. Although these activities may be useful for fostering sorting and classifying skills and helping children organize information, there is no need to delay counting-based number activities. Indeed, for children coming to school with little informal mathematical experience, it is imperative to provide ample opportunities to count and use numbers from the start.

2. Although a memorization-by-rote approach is applicable to teaching preschoolers the forward counting sequence to 12 or so, most children entering kindergarten need a very different type of training. They need to discover the patterns underlying this sequence. Counting-in-unison activities, by themselves, do not give children the opportunity or encouragement to look for patterns.

3. Activities such as counting hand claps provide practice for generating the correct counting sequence and for the one-for-one tagging process. Kindergartners, though, typically have mastered verbally counting to at least 12 and already have proficiency with one-to-one tagging. These activities do not provide practice for the keeping-track process, which is the primary cause of their enumeration errors with larger collections.

4. (a) By the time they enter kindergarten, most children naturally use counting and numbers to compare collections larger than four. The procedure of using cubes to determine whether or not two collections are equivalent (in one-to-one correspondence) is fine but may seem unfamiliar or onerous to children. That is, it may seem more abstract and cumbersome to them than using counting. (b) The Handsful Activity probably would not enhance children’s understanding of more. A preschooler could use direct perception to determine whether or not two lined up collections were equivalent or not. That is, a child will intuitively conclude that the longer row has more. This activity simply reinforces children’s intuitive tendency to base equivalence judgments on appearance. It does not help them recognize that appearances can be misleading. That two
collections are equal if put into one-to-one correspondence despite changes in appearance (number conservation). (Such an activity may be a good way to help children construct an understanding of the term less. Note that this method ties this term to the perceptual cue of length. However, most children may find it easier to check their prediction by counting.)

5. Fingers are young children’s natural medium for representing numbers and calculating with them. A program based on the philosophy of building upon children’s informal mathematics should encourage finger representation and counting. By its silence on the matter, the program reinforces the unfounded misconception held by many educators that if children are allowed to count on their fingers, they will be unable to break this bad (inefficient) habit. Research suggests that children typically invent or use more efficient methods for determining sums and differences as soon as they are developmentally ready (Baroody, 1987; Carpenter & Moser, 1984).

Key for Probe 4.4 (pages 4-17 to 4-19)

Part I. 1. Mrs. Garcia. 2. & 3. Neither focuses on the basis of Bart’s or Che’s difficulties (the need for mental images and motor plans).

Part II

1. In most cases, attributing numeral-reading and -writing difficulties to perceptual-motor deficits is questionable. If Joey had perceptual difficulties, how did he navigate his way through classrooms, hallways, and streets? If the boy suffered from perceptual confusion, why could he consistently identify reversed numerals as incorrect?

2. If Joey lacked small-muscle control, why could he make a legible 7, albeit in reverse? Kindergarten children can typically draw straight lines, make circles, or intricate designs such as \[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array} \]
Could they do so if they really lacked small-muscle control?

Part III

1. One solution is illustrated to the right.

2. Like copying numerals, this “simple” copying task is—without a “motor plan”—not so simple. In effect, like copying a numeral, success depends on knowing ahead of time how to begin, where to go next, and so forth.

Part IV

1. If a child is shown the numerals in order, he or she may simply count to determine the answer. For example, shown 1 2 3 4 5 and asked to identify the fourth numeral, a child who cannot recognize 4 might simply count "one, two, three, four" and correctly answer "four."

2. Even if a child can identify or read all normally oriented numerals correctly, it does not mean he or she has an entirely accurate mental image of the numerals. Only a partially correct mental image is necessary to identify or read correctly written numerals such as those in a textbook, on the chalkboard, or on a worksheet. For example, to distinguish a 6 from all numerals except 9, all a child needs to see is a symbol with a curve and loop. To distinguish a 6 from a 9, the child simply has to note that the loop was attached to the lower portion of the curve, not to the top. He or she does not have to know that the loop is attached to the right-hand side of the curve, which is necessary for distinguishing a 6 from a 9. To test whether or not a child has a completely accurate mental image, it is important then to include reversed numerals as well as correctly written numerals. In addition, it is important that the numerals be shown in random order.

Part V

1. The Case of Zelda Zero. Zelda had a completely accurate mental image for 2 and 4. The fact that the girl could distinguish between correct 2s and backward 2s indicates she even knows the left-right orientation of the numeral. Thus, her writing difficulties with these numerals are probably due to an incomplete or inadequate motor plan. Zelda had an incomplete mental image for 3. She knew the parts and part-whole relationships of a 3 but not its left-right orientation. Without a completely accurate mental image of 3s, she could not have a completely accurate motor plan. Zelda could not recognize or read the numerals 6 to 9, suggesting an incomplete or inaccurate mental image of these numerals. More
specifically, she probably did not know the parts or the part-whole relationships of these numerals.

**Key for Probe 4.5** (page 4-24)

**Mathematics Their Way**

1. We believe that numeral writing should be introduced purposefully.

2. Drawing the numerals in two colors may help children to recognize the parts of a numeral. Two numerals (4 and 5), though, might better be viewed as having three parts.

3. The two-color method is not an effective way of minimizing or remedying reversals because it does not help the child establish where to start and what direction to go in. In other words, it does not help children construct a motor plan. The numeral-writing training of the *Mathematics Their Way* program might be more effective if a teacher *explicitly described* his or her motor plan while modeling how to write a numeral. It may be especially important for some children to make clear where to start and what direction to head off in.

4. Unfortunately, copying numerals in the air or in the palm of a hand does not leave a record of the effort, which could signal to the teacher or children themselves that a modification in an (incorrect) motor plan is needed.

5. Tracing and copying activities may be more useful if children are encouraged to verbalize the motor plan as they work.

6. The Writing Paper does indicate where to start and what direction to move initially. Verbalizing a motor plan makes it more likely that children will internalize a complete and accurate motor plan. Finally, it is important to take practice a step beyond numeral copying: Children should be checked on their numeral-writing skill *without* any model numerals present (cf. Writing Papers Activity, p. 51, Baratta-Lorton, 1976).

**Key for Some Tips On Using Technology** (page 4-26)

**Calculators: Keystrokes**

To count by twos (Activity Card 2-1), keying $+ 2 , = 0 , = 0 , = 0 . . . \text{or } 2 , + 2 , = , = , = , = . . . \text{also works}$. To count backwards from 10 (Activity Card 2-2), key in $10 , - 1 , = ; \text{hitting the} = 10 \text{times brings the count to 0}$. (b) If the $=$ key is hit more than 10 times, the calculator continues the count with negative numbers. This may be one way of raising interest in these numbers. To count by twos starting from five (Activity Card 2-3), key in $5 , + 2 , = \text{repeatedly}$. 