FOSTERING AND EVALUATING MEANINGFUL LEARNING: MAKING CONNECTIONS AND ASSESSING UNDERSTANDING

TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 3.1: Meaningful Instruction

The aim of this unit is to help adult students understand why the NCTM (1989) puts so much emphasis on "making connections." Making connections is at the heart of meaningful and purposeful learning. Recent research indicates that remembering is enhanced when new information is connected to a web of familiar knowledge and forgetting is more likely when it is introduced in isolation. For example, remembering a person's name, is easier when one can connect it with common interests, friends, events, and so forth.

Piaget's notion of assimilation, which involves making a connection between new information and existing knowledge, is fundamentally important to a constructivist approach to mathematics instruction. It helps explain why it is important to check for developmental readiness, why it is essential that instruction build on children's informal or other existing knowledge, and why instruction may need to be adjusted in order to accommodate individual differences and children with different backgrounds.

Probe 3.1: An Assimilation Demonstration (pages 3-4 and 3-5 of the Student Guide) was designed to underscore why it is important for teachers to take into account Piaget's (1964) principle of assimilation. Although Part I is a technically correct description of the fictitious "Giant Alien Number System" (GANS for short), it is also highly verbal and abstract. That is, this description typically does not connect well with students' existing knowledge. The result is that nearly all fail to comprehend the description and are largely baffled by the subsequent assignment (Exercises 1 and 2). Part II illustrates how much more comprehensible new information can be if it is related to what students already know.

Probe 3.2: The Importance of Informal Mathematical Knowledge (page 3-7 of the Student Guide) was designed to prompt discussion about the importance of taking into account, and building on, children's informal knowledge. See the answer key for Probe 3.2 (on page 88 of this guide) for a more complete discussion of these points.

Investigation 3.1: The Fundamental Counting Property (page 3-9 of the Student Guide) illustrates how a formal (probability) procedure can be taught in a meaningful fashion by encouraging children to solve problems informally, to look for patterns or relationships, and to summarize their discovery explicitly as a rule or procedure.

Investigation 3.2: Math Detective—Using Examples and Nonexamples (page 3-17 of the
Student Guide) can help underscore that an ultimate aim of mathematics instruction is to develop an explicit understanding of concepts and procedures and the value of using examples and nonexamples to define concepts.

**Unit 3•2: Assessing Understanding**

The aims of this unit include helping adult students understand the purposes of assessment as outlined by the NCTM (1989, 1995) Standards. For example, to understand the inference standard, they need to appreciate the nature of children’s errors—that such errors are often systematic and reflect a pupil’s misconception or incomplete knowledge. (This is the aim of Part I of Probe 3.4: Reflecting on Assessment on page 3-19 of the Student Guide.) Furthermore, adult students need to appreciate that in order to accurately assess understanding, teachers need to go beyond the correctness of an answer (the product of students’ efforts) and examine how children arrived at their answers (the underlying process). This is the aim of Part II of Probe 3.4.

Probe 3.5: Techniques for Assessing Mathematical Understanding (pages 3-24 to 3-26 of the Student Guide) illustrates five ways of assessing understanding, namely rubric scoring, interviewing pupils, using open-ended questions, analyzing error, and writing story problems. Many students will not be familiar with such assessment methods.

**Unit 3•3: Integrating Instruction and Assessment**

Many educators think of assessment as distinct from instruction—as something that follows instruction and closes it out. Unit 3•3 is intended to underscore the point that assessment should be an integral aspect of instruction and an ongoing process. Vignette 1 in Part I of Probe 3.6: Integrating Assessment and Instruction (pages 3-32 to 3-35 of the Student Guide) illustrates how informal assessment can guide instruction and make it more profitable. Vignette 2 illustrates a lost opportunity for using student errors as a springboard for mathematical inquiry and learning; Vignette 3, how student errors can promote discussion, insight, and invention. Parts II and III provide "hands-on" experience with two powerful techniques that serve simultaneously to promote learning and provide insight into students’ knowledge: student-generated tests and concept mapping.

**SAMPLE LESSON PLANS**

**Project-Based Approach**

The following is a worthwhile project that simultaneously involves students in learning the content of chapter 3 and assessing their understanding of this material: Develop a comprehensive plan for assessing students’ understanding of the material in chapter 3. The plan should include major process and content goals, how each goal is measured, and specific assessment tasks. Justify your choice of tasks in terms of the assessment standards outlined by the NCTM (1995). Evaluate the diversity of your tasks (e.g., Does your plan include performance assessment, oral assessment, and written assessment? Alternatively, assign the problem Suggested Activity 7 ("Authentic Assessment") on page 82 of this guide. As a bonus project, an instructor can have students choose one of the other SUGGESTED ACTIVITIES on pages 81 and 82.

**Single-Activity Approach**

Probe 3.6: Integrating Assessment and Instruction (page 3-32 to 3-35 of the Student Guide) could provide the basis for one or two classes of instruction. In addition to underscoring the point that assessment and instruction should go hand in hand, Vignette 1 in Part I could be used to highlight they key point that the heart of meaningful instruction is making connections. In the case of David, there was a gap between the formal instruction (the symbolic representation of addition and subtraction) and the child’s informal knowledge of arithmetic. Once the teacher recognized this gap, she helped him interpret (assimilate) the new symbolism in terms of his existing (informal) knowledge.

Vignette 2 can serve to emphasize that children’s errors are often not haphazard but systematic efforts that reflect their existing knowledge. By simply focusing the correctness of answers (product) and dismissing errors, teachers miss opportunities to understand better their students’ mathematical thinking and, perhaps, to promote it. Vignette 3 not only illustrates how student errors can create cognitive conflict and discussion, it demonstrates how pupils can use the resulting insight to reinvent formal procedures. A relatively complete assessment of children would include, for example, a disposition to check out discrepancies and a willingness to admit errors. It would
also include the flexibility and persistence to adapt a correct procedure or to amend an incorrect (invented) one.

Devising a student-generated test for Unit 3•2 (Part II of Probe 3.6) would serve the twin aims of promoting student learning about the content and assessing their depth of understanding of this content. Note that this activity puts considerable responsibility for learning on the students and assesses their understanding in an open-ended way.

Part III on concept mapping begins with the relatively easy task of evaluating a partial complete concept map (Question 1) and the somewhat more difficult task of completing this partially complete map (Questions 2 and 3). Note that this latter task should facilitate student self-assessment. That is, they should quickly recognize whether or not they really understand all the geometric concepts listed and what concepts they are unsure of. Questions 4 and 5 highlight how concept mapping can result in conflict or doubt and, thus, serve as a basis for inquiry and learning. Question 6 is intended to prompt a discussion how concept mapping can be used as an assessment tool. By observing students construct a map or examining their completed map, a teacher can quickly discern how well connected their knowledge is. Question 7 can serve to underscore the importance of linking phrases in concept mapping. These phrases force students to explicitly define how concepts are related and, thus, is crucial to assessing how well connected their knowledge is.

Multiple-Activities Approach

1. Probe 3.1: An Assimilation Demonstration (pages 3-4 and 3-5 in the Student Guide) can help underscore—in a humorous and thought-provoking way—that to be meaningful, instruction must build on (connect with) what students already know. Either act out or have students read the text of Part I. Then have them complete Exercises 1 and 2. Students typically are utterly confused by the highly abstract text and frustrated by the assignment. With any luck, this will help them empathize with children who face decontextualized instruction that is unconnected to their experience. Typically, students experience far greater success on Part II of the probe, in which the instruction is related to the familiar experience of a number line. This part can serve as an interesting pattern-recognition activity and as a basis for making and evaluating conjectures. (See the key for Probe 3.1 on page 88 of this guide for a detailed explanation.)

2. Probe 3.2: The Importance of Informal Mathematical Knowledge (page 3-7 in the Student Guide) can serve to humorously highlight the importance of building on children’s informal knowledge in order to foster mathematical power. A frame-by-frame analysis or discussion can help to explicitly make a number of key points. For example, Frames 1 and 2 illustrate the confidence children have when they can apply their informal knowledge to school-math tasks. Frame 3 exemplifies a key limitation of informal knowledge, namely difficulty working with big number. Frames 3, 4, and 5 illustrate the helplessness children often experience when school math is beyond their informal knowledge. Frame 6 again contrasts the sense of power children enjoy when they can use their informal knowledge.

3. Investigation 3.1: The Fundamental Counting Property (page 3-9 of the Student Guide) illustrate how children can be helped to rediscover a probability procedure by building on their informal knowledge. In Part I, encourage students to solve Combination Problems A and B informally (e.g., by making a drawing, a table, or a list). By completing the table in Part I, they should see how children could quickly discover the shortcut for determining the number of combinations in problems like problems A and B: multiply the number of possibilities in each collection. By repeating this procedure with Part II, they should also see how children can discover the shortcut for determining the probability of two independent events occurring together: multiply the independent probabilities. With any luck, adult students will appreciate better that procedures do not have to be taught as math magic (incomprehensible procedures that are simply memorized by rote) but can be rediscovered in a meaningful fashion.

4. Investigation 3.2: Math Detective—Using Examples and Nonexamples (page 3-17 of Student Guide) illustrates how examples and nonexamples can help children induce the critical attributes of a concept and define it explicitly. For Activity I, an instructor can read a question, have the students use what they previously learned to answer it, and then (using the answer key on page 89 of this guide) provide feedback. Then repeat this process for each successive question until the concept (an arithmetic sequence) is defined. To better model
an inquiry-based approach, an instructor can withhold feedback and can have the students discuss their answers to each item, debate their different opinions, and—if necessary—vote on the "correct answer." If the debate becomes deadlocked, the instructor can prompt the class to consider how the conflict can be resolved (e.g., look up arithmetic sequence in a dictionary). Activity II illustrates how using best example—a wide variety of examples and nonexamples that clearly define all the critical attributes—can provide a relatively direct approach to teaching a concept.

5. Part I of Probe 3.4: Reflecting on Assessment (page 3-19 of the Student Guide) can help to make a number of the points: (a) Assessment should serve to plan or adjust instruction, not merely to assign grades; (b) Errors are a natural part of learning, not a sign of stupidity or laziness; (c) Errors are frequently systematic, not merely careless and random. Encourage students to analyze the examples of systematic errors. Cassie and Deborah's errors are due to procedures these children invented because they could not accurately remember the school-taught algorithm. Pam's error is due to an incompletely-learned algorithm. (See page 3-20 of the Student Guide for a more complete description of these systematic errors). Part II of the probe can help underscore the key point that teachers need to examine process. Encourage students to examine each case described in Question 1. Have them decide whether or not the incorrect answer clearly indicates that the child described does not understand the concept tested. In each case, adult students should discover that a teacher could not be sure if they only consider a child's answer. (See the answer key for Probe 3.4 on page 89 of this guide for a more complete discussion of this point.)

6. Completing selected portions of Probe 3.6: Integrating Assessment and Instruction (pages 3-32 to 3-35 of the Student Guide) can help reinforce the point that assessment can and should be an integral aspect of instruction. Vignette 2 in Part I illustrates how judgmental feedback can quash mathematic inquiry and undermine children's mathematical power. If instead of dismissing errors, the teacher had asked students to justify their answers, everyone may have benefited. Vignette 3 illustrates how errors or perceived errors can serve as a springboard to inquiry and discovery. Doing selected portions of Part III can introduce students to concept mapping. Because creating a concept map involves explicitly thinking about and specifying how concepts are defined and related, it can be an invaluable tool in prompting discussion and learning. It can also provide teachers invaluable assessment information about what connections students have made, their incorrect connections, and what connections they do not see.

SAMPLE HOMEWORK ASSIGNMENT
Read: chapter 3 of the Student Guide.

Study Group:

- Questions to Check Understanding: 1, 2, 3, 4, 7, and 8 (pages 84 to 86 of this guide).
- Suggested Activity: 5 (page 82).
- Problem: Bogus Bill (page 87).
- Bonus Problem: An Elimination Tournament (page 87).

Individual Journals: Writing or Journal Assignment 1 (page 86).

FOR FURTHER EXPLORATION

QUESTIONS TO CONSIDER

1. (a) How are arithmetic sequences and patterns related? Are all arithmetic sequences patterns? Is the reverse true? (b) Draw a concept map that illustrates the relationship between arithmetic sequences and patterns. Include in your map the following two subtypes of patterns: repeating patterns (i.e., patterns in which a core keeps repeating, such as ABCABCABC ... or .148514851485) and growing patterns (i.e., patterns in which a core keeps expanding, such as AABABCABCD... or .1121231234...). Consider the following questions: Is an arithmetic sequence a repeating pattern, a growing pattern, or some other kind of pattern (see Box 2.1 on page 44 of this guide for examples)? That is, does it meet the definition of a repeating pattern or a growing pattern? Is a geometric sequence a repeating pattern, a growing pattern, or something else? If an example fits the something-else category, where should it be placed in the concept map?
2. Systematic errors often arise when children do not comprehend a new task and try to use a familiar procedure to determine an answer. Examine the evidence in the case of Bryan. What "fix" did this boy make? Bryan could compare fractions with a common denominator such as \( \frac{3}{5} \) vs. \( \frac{4}{5} \), \( \frac{5}{6} \) vs. \( \frac{7}{6} \), \( \frac{3}{4} \) vs. \( \frac{2}{3} \), and \( \frac{3}{7} \) vs. \( \frac{2}{7} \). That is, he correctly identified the larger fraction. However, when Bryan encountered items such as \( \frac{3}{2} \) vs. \( \frac{3}{4} \), \( \frac{4}{5} \) vs. \( \frac{4}{9} \), and \( \frac{1}{3} \) vs. \( \frac{1}{5} \), he was consistently incorrect. He also consistently chose the smaller fraction for comparisons such as \( \frac{1}{3} \) vs. \( \frac{3}{10} \), \( \frac{2}{5} \) vs. \( \frac{3}{2} \), and \( \frac{2}{3} \) vs. \( \frac{3}{5} \). For items such as \( \frac{3}{4} \) vs. \( \frac{4}{5} \), \( \frac{2}{3} \) vs. \( \frac{5}{7} \), and \( \frac{9}{10} \) vs. \( \frac{7}{8} \), Bryan was correct.

3. Consider the following journal entry of pre-service teacher: I think the most negative feelings I got were from teachers who graded solely on what my answer was. I would spend 4 hours working on one math assignment and get a C because even though I completed the problems, they weren't all correct. I understand that part of math is getting the correct results, but often there is never a reward for trying strategies for new or different problems—Lanie Diasar. How can a teacher assess problem solving in a fair manner—in a way that would take into account Lanie's concern?

**SUGGESTED ACTIVITIES**

1. Choose at least two concepts that would be appropriate for the grade level you plan to teach. (a) Design an activity that uses examples and nonexamples in an open-ended fashion (like Activity I of Investigation 3.2 on page 3-17 of the *Student Guide*) to help students discover the critical attributes of a concept. Identify questions that might prompt debate—where the examples and nonexamples do not clearly indicate the correct answers. (b) Design an activity that uses best examples and a wide variety of nonexamples (like Activity II of Investigation 3.2) to help students induce the critical attributes of a concept.

2. Watch the videotapes "Overview: Understanding and Manipulatives" and "Introduction to Clinical Interviews" from the *Children's Mathematical Thinking: Videotape Workshops for Educators* by H. P. Ginsburg and colleagues and published by Everyday Learning Corporation. (a) Answer the questions posed on the videotapes. (b) Conduct at least one interview illustrated on these videotapes with at least two children. (c) Write a report on the interview, including what you learned about the child, what else would be helpful to learn about the child, instructional recommendations, and how your interview could have been improved. (d) Write a report on how the videotapes changed your views about children's mathematical learning, using manipulatives, and assessing children's knowledge.

3. Examine the *Hot Dog Stand: The Works* (see page 3-40 of the *Student Guide*) or other educational software intended to build connections. Evaluate the extent to which the program connects (a) mathematics to everyday situations (i.e., is realistic), (b) mathematical topics to each other, (c) formal mathematics to informal knowledge, (d) formal procedures to conceptual knowledge, and (e) mathematics to other subject areas.

4. Examine a textbook lesson, a lesson described in a teacher resource, a curriculum guide, or a web site and evaluate the extent to which it might help build the connections described in the previous question.

5. (a) Using the General Rubric illustrated on page 3-24 of the *Student Guide*, design a task-specific rubric for a mathematical concept of your choice. (b) Devise a general rubric for assessing procedural knowledge. Then use it to devise a task-specific rubric for knowledge of a particular procedure.

6. (a) Design a close-ended performance assessment item for a concept or procedure and a grade level of your choice. Describe your scoring protocol—that is, what a teacher would look for that would indicate different levels of understanding. (b) Do the same, but this time design an open-ended item.

7. (a) Design an authentic assessment for the grade level and mathematical areas of your choice. (b) Briefly justify how your assessment item meets the criteria for this type of assessment noted in the *Student Guide*. (c) Briefly describe what process goals (e.g., problem-
solving competencies), content-knowledge goals (e.g., mathematical concepts and connections), and affective goals (e.g., beliefs or dispositions) your item might tap.

8. (a) Devise a general checklist of processes, understandings, and dispositions that you feel are important for children at the grade level you intend to teach. (b) Devise a specific checklist for a particular topic that would be covered at this grade level.

9. Try out your interviewing skills with a younger relative, neighbor child, lab-school student, or some other elementary-level child. Select a problem or other task from one of the chapters in this book (e.g., for a 5-year-old, read a simple addition word problem from chapter 5). Present the task and watch how the child tackles it. As needed, ask the child follow-up questions to determine how he or she determined the answer.

10. Videotape an interview by a news reporter or talk-show host. Analyze the interview in terms of the guidelines outlined in Probe 3.5 on pages 3-24 and 3-25 of the Student Guide (e.g., Did the interviewer use leading questions or put words in the mouth of the interviewee?).

11. Using the protocol described in Box 3.1 on the next page, administer the number-conservation task (Piaget, 1965) to two 4-year-olds and two 5-year-olds. If audiotape or videotape equipment is available, record the interviews. Follow the procedure and answer the questions delineated in Box 3.1.

12. Conduct a case study of a child having mathematical learning difficulties. Formulate an agreement with your instructor as to the duration of the case study and the length of your report. (a) Locate or devise tasks that might be helpful in identifying the child’s specific informal and formal strengths and weaknesses (see, e.g., the references listed on pages 3-38 and 3-39 of the Student Guide). (b) Interview the child using the tasks and, if need be, follow-up probes (questions). Summarize your findings in a report. (c) Indicate in your report suggestions for remediating the child’s difficulties. (d) Implement remedial instruction with the child and evaluate the success of your efforts. Write a report that describes the instruction, what seem to be helpful and what was not, and the extent of the child’s progress.

13. After reading the section on Oral Assessment (page 3-30 in the Student Guide), visit a classroom and evaluate the teacher’s use of student questioning, student justifications, and peer-group exchanges as tools for assessment. (b) Devise a lesson and indicate how you could use these oral-assessment techniques to gauge student knowledge and learning. (c) Implement your lesson in an elementary-level class. Indicate what you learned about the children from using oral-assessment techniques. Evaluate your use of those techniques.

14. Keep a journal that describes your reflections about each class meeting, including your feelings; insights about mathematics, mathematics teaching, or children; questions or difficulties; and self-assessment about your progress.

15. Using Inspiration Software® (© 1994 by Inspiration Software® Inc., Portland, Oregon), Sem Net (described on page 3-40 of the Student Guide), or some other drawing program, create a concept map that summarizes your understanding of Unit 3•2 or some other unit.

16. Stories can provide an opportunity for introducing, discussing, or practicing mathematical content. (a) Write two stories to address the mathematical (and other) content of your choice. The story in Box 3.2 on page 83 would be suitable for upper elementary students. Note that the story involves a rate problem and fraction arithmetic. It could also serve to introduce vocabulary such as dawdle, tardiness, vexed, and steely. (b) Compile a list of children’s literature books that contain mathematical content at the grade level you plan to teach. Use a computer file or index cards. You may wish to cross-reference entries by title, author and mathematical content. Plan how you could use these opportunities to introduce, discuss, or practice mathematics skills in concepts. (c) Examine the basal reader or set of stories used in the reading program at the grade level you plan to teach. Note what mathematical content is mentioned and how you might use the references to explore mathematics.
Box 3.1: Protocol for a Standard Interview to Assess Number Conservation (Piaget, 1965)

- Prepare for the interview ahead of time. Familiarize yourself with the task protocol and videotape equipment. Also assemble the following task materials: (a) seven white chips or counters, (b) ten blue chips or counters, and (c) a Cookie-Monster muppet.

- After a familiarization period to get the child comfortable, start the videotape.

  - Put out seven white chips in a row about 1/2-inch apart. Give a child 10 blue chips and say, "PLEASE PUT OUT THE SAME NUMBER OF BLUE COOKIES (chips) AS WHITE COOKIES." If necessary, help the child create the initial 1-to-1 correspondence.

  - After the 1-to-1 correspondence has been established ask the standard question (SQ): "COOKIE MONSTER WOULD LIKE TO KNOW—ARE THERE AS MANY BLUE COOKIES AS WHITE COOKIES OR DOES ONE ROW HAVE MORE?" If the child does not appreciate the initial equivalence, test another child.

  - Then—while the child watches—spread out the white row so that each end item clearly extends beyond the end item of the blue row. Again ask the SQ.

- Next, check resistance to counter-suggestion. (a) If a child conserves (answers correctly), say, "COOKIE MONSTER LOOKS AT THIS AND SEES THAT THE WHITE ROW OF COOKIES IS LONGER, AND HE THINKS THAT MAYBE THERE ARE MORE WHITE COOKIES." (b) If a child does not conserve (answers incorrectly), say, "BEFORE WE PUT ONE BLUE COOKIE WITH EACH WHITE COOKIE AND THERE WERE THE SAME NUMBER, AND COOKIE MONSTER STILL THINKS THERE'S THE SAME NUMBER IN EACH ROW. WHAT DO YOU THINK?"

- Finally, ask conservers to justify their responses. Acceptable justifications include: (a) identity ("They're the same, you didn't put any more in"); (b) compensation ("They're the same, there's more space between these rows"); (c) reversibility ("If we put them back they'd be all lined up again and there wouldn't be more white ones.").

- Score each child's response according to the following criteria:
  0 = Does not recognize the initial equivalence (that two aligned rows of seven chips are equal).
  1 = Constructs initial one-to-one correspondence but unable to conserve (i.e., after one row has been lengthened, indicates that a row has more chips).
  2 = Responds inconsistently (e.g., conserves but changes his/her answer after a counter-suggestion).
  3 = Conserves and resists counter-suggestion after counting to determine answer.
  4 = Conserves, resists counter-suggestion, but doesn't give an adequate justification.
  5 = Conserves, resists counter-suggestion, and gives an adequate justification.

Consider the following questions: (a) Why is the preparation for the interview (e.g., practicing the protocol and collecting the materials) important? (b) Why are poker chips or counters (items with identical sides) used rather than coins or checkers? (c) Why is a muppet used? (d) Did each of your children clearly fall into one of the scoring categories? What conclusions can you draw about each child's understanding of number relations? Defend your position. (e) What advantages did this standard interview task afford? What were the disadvantages? (f) What kinds of problems did you encounter during the interview? What did you learn about interviewing children?

Box 3.2: Dudley the Dawdler

Dudley was late for school again, arriving just in time for lunch. When Miss Fern inquired impatiently if he had been dawdling again, Dudley was pretty sure what she meant. However, to put Miss Fern off the track, he inquired in a most earnest manner, "What does dawdle mean, Miss Fern? It sounds like a most interesting word, and I surely would like to add it to my vocabulary."

"To move slowly," said Miss Fern softly, just a bit surprised by Dudley's sudden interest in the language arts. "Did you walk to school slowly or is there some other reason for your tardiness?"

"Slowly, Miss Fern? Could you be more precise?" replied Dudley.

Vexed, Miss Fern asked in a steely tone, "Dudley, how far is it from your house to the school?"

"One quarter of a mile," answered Dudley cheerfully, not recognizing the trap that Miss Fern was setting.

"And how long did it take you to walk to school?" she pressed.

"Oh, a half an hour," replied a still unsuspecting Dudley.

"Tell me Dudley," inquired Miss Fern sweetly, "how fast were you walking then?"

Dudley finally recognized that he had a formidable and worthy adversary in Miss Fern.

What arithmetic operation should Dudley use to figure out his rate? What was his rate? If casually walking a mile required 30 minutes, how much faster or slower would this pace be than Dudley's rate?
HOMEWORK OR ASSESSMENT

QUESTIONS TO CHECK UNDERSTANDING

1. Answer the following true-false questions. Circle the letter of any statements that, according to the Student Guide, is true.

   a. Primary-level (K-3) mathematics lessons should not be integrated with other subject matter because children are too cognitively immature and may become confused.

   b. Formal instruction should build on children's informal knowledge.

   c. The symbolic phase of instruction (working exclusively with symbols) should precede the connecting phase (linking written symbols or procedures to concrete models and examples).

   d. Children are generally adept at spontaneously seeing the connection between written mathematics and concrete models.

   e. Using manipulatives guarantees that children will understand a mathematics lesson.

   f. Ideally, encouraging children to devise or learn manipulative-based models should precede and lead to an initial conceptual understanding of a topic, not the reverse.

   g. Children should be encouraged to invent concrete procedure but not written ones, because their invented written procedures are very seldom as efficient as school-taught algorithms.

   h. To underscore the abstract, symbolic nature of mathematics, it is important to introduce concepts by showing children the relevant written representation.

   i. Written arithmetic should be made to look like a simpler and faster way of doing what children already know how to do.

   j. In general, even an effective teacher cannot expect to promote meaningful learning quickly.

   k. Teacher observation, listening, and questioning should be the primary basis for assessing children, not written tests.

   l. Concept mapping is useful as an assessment tool but not as a teaching/learning tool.

   m. Miss Brill had her class practice multiplying fractions with one manipulative (Fraction Tiles) but then tested their understanding with a new manipulative (Fraction Circles). Using a new manipulative is an unfair test of understanding.

   n. A clear implication of the mathematics and coherence standards is that a national assessment test is necessary.

   o. According to the equity standard, children's assessment should be done strictly on merit and, thus, not take into account children's backgrounds.

   p. According to the openness standard, students should be informed about what they are expected to know and how they will be tested before assessment takes place.

   q. According to the inference standard, when assessing students' knowledge, teachers should avoid inferences, which are subjective, and use direct (objective) measures of their knowledge.

2. Circle the letter of any statement that—according to the Student Guide—is true. Change the underlined portion of any false statement to make it true.

   a. Informal mathematical knowledge is learned largely through manipulative-based school instruction.

   b. Informal mathematical knowledge provides a basis for assimilating formal instruction at the primary-level but not intermediate level.

   c. A key reason for learning difficulties is a gap between the teacher's instruction and textbook assignments.

   d. It follows from the principle of assimilation that meaningful learning is most likely to occur when information is moderately novel.
e. It follows from the principle of assimilation that children are most attentive to lessons that involve highly unfamiliar information.

f. Using a variety of examples and nonexamples can be a helpful tool in implicitly inducing the critical attributes but not explicitly defining them.

g. In the skills approach, teachers use manipulatives to help explain or justify a procedure, whereas in the conceptual approach, children are encouraged to use manipulatives to invent their own strategies or to justify their position.

h. Standardized tests are inadequate to diagnose difficulties because they fail to focus on process.

i. Assessment should focus on product (the answers children produce).

j. In general, performance assessment or oral assessment is more likely to reveal process information than is written assessment.

3. Mathematical understanding can best be characterized as (circle the letter of one of the following): (a) a sudden insight; (b) an intuition; (c) a connection; (d) an ablation; or (e) a deduction

4. The principle of assimilation is fundamental to a teacher’s understanding of learning. Circle the letter of any conclusion that follows from this fundamental principle.

a. Meaningful learning is most likely to occur when information is moderately novel.

b. Children are most attentive when the information presented to them is highly familiar.

c. Children have a limited attention span and can take in new information for only a short period of time.

d. Information not related to existing knowledge cannot be understood.

5. Analyze the following four vignettes. (a) For each, decide whether or not the principle of assimilation applies. (b) Indicate which vignette(s) illustrate(s) Piaget’s moderate novelty principle. Briefly justify your choices.

A. Using fingers to represent six and one more, Alexi determined the sum by quickly counting the fingers. Missing the sixth finger he noted, “Six, pooh.” He then spontaneously recounted carefully and answered seven.

B. Unfamiliar with the representation $4^2$, Byron asked what it meant. A group mate pointed out that it meant two fours multiplied—four times four. Byron was then able to understand the notation $5^3$.

C. Miss Brill found, what seemed to her, an interesting problem. Unable to understand the problem, most students in the class quickly lost interest in the task and began amusing themselves in uneducational ways.

D. Edina asked her teacher what triangle meant. Miss Lilly responded by asking her what tri meant. When Edina indicated she did not know, the teacher wrote out a number of words such as tricycle, triplets, triple, asked what they meant and asked what they had in common. "Oh, so tri means three," concluded Edina.

E. Juanita was puzzled by the funny symbol she found while leafing through her math book. "What's this?" she asked her teacher, pointing at a ÷ symbol. Miss Brill replied: "That's the symbol for division. Division is like when you have 12 candies and 4 people and you want to know how many candies each person will get." "Oh, I get it now," said Juanita gleefully.

F. George listened carefully to his teacher as she explained the renaming (borrowing) algorithms for multidigit subtraction. He had no idea what she meant by renaming and why it would be done. George stopped listening.

6. Assume a child does not understand the underlying concept of a test question. (a) What is the probability of a false success on a multiple-choice item with four choices? (b) What is the probability of a false success on a modified multiple-choice item with four choices? (With such an item, a student may select more than one choice or even all the choices. Assume that at least one choice must be selected.)
7. For each of the following cases, indicate whether it represents a true success, a false success, a true shortfall, or a false shortfall.

a. On a test, a child correctly determines that the quotient of $\frac{1}{2} \div \frac{1}{8}$ is 4 but cannot explain why this answer is larger than $\frac{1}{2}$.

b. Adrianne's work on a test item is shown below.

\[
\begin{array}{c|c|c|c}
\text{Subtract:} & 57 & -18 & 39 \\
\hline
49 & \text{ } & \text{ } \\
\end{array}
\]

(c) not given

b. Adrianne's work on a test item is shown below.

\[
\begin{array}{c|c|c|c}
\text{Subtract:} & 57 & -18 & 39 \\
\hline
49 & \text{ } & \text{ } \\
\end{array}
\]

(c) not given

c. D'ray, who did not understand place value or renaming, recorded an answer of 49 for the test item above.

d. A student responded to the following word problem with an answer of 20, explaining "of means multiply, so I multiplied four times five and got 20 percent!"†

8. Evaluate the organization of the concept maps shown below. The children were asked to represent the following concepts: centimeter, English, inch, metric, and units of length.

A Dental Survey (4-6). Four of five dentists interviewed recommended Yukkey Gum. What percentage of the dentists interviewed did not recommend it?

9. Construct a hierarchical concept map that includes the following concepts: (a) analyzing homework errors, (b) asking for student justifications, (c) assessment methods, (d) authentic assessment, (e) checking students' answers written on a minislate, (f) informal observations, (g) journals, (h) listening to debates or panel discussions, (i) listening to peer-group exchanges, (j) methods for assessing process, (k) open-ended test items, (l) oral assessment, (m) performance assessment, (n) questioning students, (o) student portfolios, (p) translating a symbolic equation into a story problem, and (q) written assessment.

WRITING OR JOURNAL ASSIGNMENTS

1. Assess your progress in the course, including your problem-solving ability. What material covered in the Student Guide or class, if any, puzzles you?

2. According to the Student Guide, as children's knowledge becomes more well connected, their reasoning and problem solving became more flexible and effective and their ability to assimilate new material increases. Briefly explain why this is the case.

3. Chapter 2 of the Student Guide distinguished among three types of tasks: exercises, problems, and enigmas. How would the moderate novelty principle predict children would respond to each of these tasks? Justify your answers.

4. You intend to assess your students' knowledge using the recommendations outlined in this guide. How could you help parents, other teachers, or your principal understand (a) the key differences between this new approach and the traditional approach to assessment, and (b) why you want to use the former and not the latter.

5. To help your students order fractions (e.g., determine whether $\frac{2}{3}$ or $\frac{3}{4}$ is larger), you encouraged them to work with one manipulative (Fraction Circles). To assess their understanding of fraction ordering, you tested them with another (Cuisenaire rods). Some students complained this was not fair. Do they have a case? Why or why not? How can you respond to their complaint?

6. Over a period of several months, you have used concept-mapping activities on a number of occasions. When you ask your class to construct a concept map on percent, several students grumble, "I hate concept maps. They're so hard." How can you respond to help these students see the value of doing a concept map?

7. What do you say to a colleague to convince her that (a) student analysis of errors, (b) student input, (c) student-generated test, and (d) concept mapping are valuable means of assessment?

8. Briefly explain how the approach for evaluating children's knowledge recommended in the Student Guide is different from the traditional approach to evaluation.

9. Whenever Miss Brill asked her class a question, LeMar, an outstanding mathematics student, invariably answered quickly and correctly—before the rest of the class had even begun to digest the question. She noticed that the other students resented him for this. At first, this was expressed subtly via eye rolls and sighs. Then the class began belittling LeMar in more open and hostile ways, calling him tech or egghead. LeMar pretended not to be bothered by the slights. One day, though, LeMar turned on a tormentor with a vicious attack, "It's better than being stung by the dumb bug. In your case Rodney, a whole hive of dumb bugs." If the tension were not bad enough, Miss Brill noticed that the rest of the class was again becoming passive. Where students were once beginning to participate actively in class, now they simply turned their heads toward LeMar and waited for the answer. How could Miss Brill get the rest of the class actively involved in answering questions, without discouraging the highly capable LeMar?

10. (a) Evaluate Hillary's response to Question 30 below. (b) Evaluate the test item as a measure of the concept of perimeter.

30. How far around?

   (✓) 11
   ( ) 9
   ( ) 6
   ( ) not given

PROBLEMS

■ An Elimination Tournament (♠ 5-8)

Eight teams are seeded 1 to 8 for the first round of an elimination tournament (the loser of a game is eliminated from the tournament). The highest seed always plays the lowest seed. In the first round, seed 1 plays seed 8; seed 2 plays seed 7, seed 3 plays seed 6, and seed 4 plays seed 5. If, for example, seeds 1, 2, 6, and 5 won in the first round, then seed 1 would play seed 6 and seed 2 would play seed 5 in the second round. (a) Taking into account that there are other second round possibilities, how likely is it that the team seeded first will play the team seeded second or third in the second round: certain, highly likely, possible, or impossible? Justify your answer. (b) Excluding the team they defeated in the first round, is there a seed that could wind up playing any other seed in the second round? Justify your answer.

■ Bogus Bill† (♠ 5-8)

Andre went into a hobby shop, bought $5 worth of hobby supplies, and paid with a counterfeit $20 bill. Not recognizing that the bill was counterfeit and not having change for the bill, the shop owner ran next door to the bank, exchanged the bogus bill for four $5 bills, and gave Andre $15 in change and the $5 worth of hobby supplies. Moments after Andre disappeared, the banker appeared in the hobby shop to inform the owner that the $20 bill he had exchanged was counterfeit. The shop owner exchanged the bogus bill for a genuine $20 bill. Later that day, the FBI appeared and confiscated the counterfeit $20 bill from the shop owner. How much did the shop owner lose? Why?

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† Based on a problem that appears in Sobel & Matelesky (1988).
ANSWER KEY for Student Guide

Key for Probe 3.1: Questions for Reflection (pages 3-4 and 3-5)

1. Using their knowledge of number lines, students might infer that the GANS number line starts with 0. Another indication that \( \varphi = 0 \) is that, like 10 in our number system, this symbol is the right-hand digit of the first two-digit GANS number. Note that the symbol for one is one point; that for two, a line (which connects two points); the symbol for three, a carat (which connects three points); that for four, a backward Z (which connects four points); and the symbol for five, a figure with five points. Note that the symbols for 6 (5 + 1) is simply combinations of the symbol for 5 and the symbol for 1. Likewise, the symbols for 7, 8, and 9 are combinations of the symbol for 5 and the symbol for 2, 3, and 4, respectively.

2. (a) Assimilation involves connecting new information to existing knowledge. (b) New information that is not connected to existing knowledge cannot be understood. The best that can be hoped for, then, is routine expertise, which cannot be applied or adapted to new learning tasks or problems.

Key for Probe 3.2 (page 3-7)

1. Note that when Peppermint Patty connected the formal expression 7 + 3 to her informal knowledge, she had no difficulty determining the answer.

2. (a) When children are unable to use their informal knowledge (e.g., when the numbers are too large for informal methods), they usually feel lost and may respond in apparently senseless ways (e.g., Peppermint Patty’s strategy of answering, "One million"). When formal arithmetic is not connected to their informal knowledge, it may seem like a foreign language to children. This undermines their sense of mathematical power or confidence. (b) In contrast, in the last frame of Figure 3.2, Peppermint Patty encounters symbols she can relate to her informal knowledge and regains interest and confidence.

3. Teachers who appreciate the power of children’s informal knowledge may be more inclined to encourage autonomous efforts and informal strategies and less inclined to spoon-feed children a mathematics curriculum and discourage their initiatives. That is, they may be more willing to serve as a guide on the side instead of a sage on the stage.

Key for Investigation 3.1 (page 3-9)

Part I

The shortcut for determining the total number of combinations is multiplying the number of choices for each item.

Part II

1. Note that the probability of choosing the green blouse is \( \frac{1}{3} \) and the probability of choosing the black skirt is \( \frac{1}{2} \). The table below shows six different outfit combinations—only one of which (G-B) is the one desired by Katarina. Thus, the probability of choosing both is \( \frac{1}{6} \).

<table>
<thead>
<tr>
<th>Blouse</th>
<th>R</th>
<th>Y</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skirt</td>
<td>b</td>
<td>Rb</td>
<td>Yb</td>
</tr>
<tr>
<td></td>
<td>w</td>
<td>Rw</td>
<td>Yw</td>
</tr>
</tbody>
</table>

2. By noting the probabilities involved and the solutions to the various problems in this investigation (see the table below), students should be able to discover the Fundamental Counting Principle: If one event can occur in \( a \) ways and another event can occur in \( b \) ways, then the total number of ways the two events can occur together or in succession is \( a \times b \).

<table>
<thead>
<tr>
<th></th>
<th>Probability of choosing first item</th>
<th>Probability of choosing second item</th>
<th>Probability of choosing both items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chancy Dresser</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>Revised</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{15} )</td>
</tr>
<tr>
<td>Part II</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{24} )</td>
</tr>
</tbody>
</table>

Key for Probe 3.3 (page 3-10)

Part II

2. Even without understanding the model, many students can use it to determine the quotient of 1.82 ÷ 1.3. This expression is sufficiently
similar to the example that they can blindly follow the steps outlined in the example. However, without understanding the model's rationale, most students probably will not be successful with the other expressions. Without a conceptual framework, even moderately novel expressions, such as 1.82 ÷ 2.6 and 1.82 ÷ 0.2, may seem impossible to answer.

Key for Investigation 3.2 (page 3-17)

Activity I

10. An arithmetic sequence involves a series of numbers formed by adding a constant amount.
11. Yes—a constant amount of -3 is added each time.

Questions for Reflection

2. (a) The previous examples and nonexamples were not sufficient to determine who was correct. Kelli’s conjecture is consistent with all the examples and, in fact, matches the mathematical definitions of an arithmetic sequence. However, Evelyn has a strong case because, to that point, all the examples are in ascending order and all the nonexamples are not. Note that Evelyn would not have had a case if something like 1, 4, 8, 13, 19 . . . had been identified as a nonexample. (b) To resolve the debate, Mr. Brandon could, for example, encourage students to look up the definition of arithmetic sequences, agree to provide examples and nonexamples that clarify the issue (e.g., 1, 4, 8, 13, 19 . . . is not an example), or simply note that 1, 3, 6, 10, 15 does not fit the conventional definition of an arithmetic sequence.

Key for Probe 3.4 (page 3-19)

Part II

1. (b) Although answers of 55 and 693 to the Pet Problem appear ridiculous, they—along with 33—are plausible answers. LeMar may have reasoned that humans are members of the animal kingdom. So, 33 pets, plus 21 students, plus a teacher totals 55. Judi may have interpreted the problem as: 21 students each brought 6 gerbils, 2 mice, 7 fish, 3 birds, 4 hamsters, 5 dogs, and 6 cats. Note that Judi’s interpretation is consistent with what is implied by the grammatical construction of the question. (c) Arturo may have written a backward 6 or an upside down 9. Without asking him, or otherwise checking how he came up with his answer, a teacher could not tell whether he is right or wrong. (d) Betty may understand the concept of first but think that the duck on the left is first, because A is the first letter in the alphabet or because she is accustomed to reading from left to right. On the other hand, she may have missed the item because she did not really understand the concept.