TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 16•1: Prealgebra and Algebra

For many adult students, algebra instruction consisted of learning rules for manipulating letters and numbers without rhyme or reason. The abstract symbols and procedures were typically not connected to prior knowledge or everyday life and, thus, were largely incomprehensible. In brief, algebra instruction has traditionally been nonpurposeful and meaningless. Like Ms. Brill described on page 16-1 of the Student Guide, many prospective and practicing teachers disliked or struggled with algebra and can't imagine teaching the topic in the elementary grades.

Unit 16•1 has three main aims: Help readers (a) appreciate the uses or importance of algebra, (b) relate algebraic symbol and manipulations to their existing and meaningful knowledge, and (c) understand how algebra can be taught in a purposeful, meaningful, and inquiry-based manner in the elementary grades. Investigation 16.1: Some Problems to Consider (pages 16-4 to 16-6 of the Student Guide) was designed to help achieve all three of these aims.

For example, Problem 2: Temperature Conversions can illustrate a key purpose of algebra (serve as a shorthand for summarizing arithmetic patterns or relations), how algebra instruction can build on what students know (grow out of meaningful arithmetic experiences), and how the investigative approach can be useful in teaching algebra (how using algebra can grow out of a purposeful, meaningful, and inquiry-based project, such as the need to convert Fahrenheit readings to Celsius readings in order to report real data on the world wide web). (For more details on using Problem 2 as a basis for class instruction, see pages 463 and 464).

Other problems in Investigation 16.1 (e.g., Problem 5 and, particularly, Problem 6) can serve to illustrate another key purpose of algebra, namely, to facilitate problem solving. Problem 5: Chickens and Pigs is a problem that can be solved in many ways. See Figure 16.3 on page 16-8 of the Student Guide for four methods. In contrast to the other methods depicted in this figure, though, algebra allows one to solve the problem relatively quickly. Problem 6: Plain and Sugar-Coated Donuts illustrates a problem that would be difficult to solve without algebra. Investigation 16.2: A Science Experiment (page 16-9 of the Student Guide) illustrates how informal algebraic strategies (try-and-adjust substitution) or formal algebraic procedures can solve a realistic problem.

The next four reader inquiries in Unit 16•1 were designed, in one way or another, (a) to foster an algebraic sense (meaningful understanding of algebraic symbols and procedures) and (b) to illustrate how a teacher can do the same with elementary-level students. Probe 16.1: Uses of Letters (page 16-11 of the Student Guide) was designed to help adult students explicitly recognize and distinguish among the four types of variables.

Probe 16.2: Algebraic Symbol Sense (pages 16-13 and 16-14 of the Student Guide) includes Clement, Lochhead, and Monk’s (1981) famous problem in order to illustrate why algebra requires adaptive expertise, not routine expertise. Even some college-level students translate “there are six times as many students as professors” into 6S = P. This illustrates a common problem in algebra: not recognizing that an equation must represent an equality. It can also provide a basis for discussing the value of the powerful balance-beam analogy: What is represented on one side of the equal sign must be balanced by (is equivalent to) what is put on the other side. (See page 464 for further discussion of how an instructor can use this problem.)

Probe 16.2 can also can help underscore the point that students of all ages can draw on their
existing arithmetic knowledge to figure out algebraic notations and manipulations. For example, students can make sense of the expression \( \frac{2n + 6}{2} \) by relating it to their divvy-up meaning of division: Double some number plus another six shared fairly between two people means each person would get some number plus three \((n + 3, \text{not} \ n + 6 \text{ or } 2n + 3)\). Probe 16.2 also introduces the useful manipulative of Algebra Tiles to help model algebraic expressions. For instance, \( \frac{2n + 6}{2} \) can be represented by two short blue rods and six yellow rods divided into two equal groups: one short blue rod and three yellow rods each.

**Investigation 16.3: Using a Math Balance to Solve Algebraic Equations Informally** (page 16-17 of the Student Guide) extends the balance-beam analogy introduced in Probe 16.2 to solving algebraic equations. It illustrates that such a concrete model and analogy can not only help children understand the *perform-the-same-operation-on-both-sides-of-an-equation* rule but can enable them to reinvent the procedure themselves.

**Investigation 16.4: Using Algebra Tiles to Solve Algebra Problems & Equations Concretely** (page 16-18 of the Student Guide) illustrates how an area model and Algebra Tiles (a manipulative introduced earlier in Probe 16.2) can be used to make sense of multiplying polynomials. This investigation also illustrates how they can enable students to rediscover the FOIL shortcut and to make sense of factoring polynomials.

**Unit 16•2: Functions**

Although most adult students may have an intuitive sense that functions involve a mathematical rule, they may not have a clear understanding of the concept, why it is an important topic for elementary instruction, or how the topic can be introduced to children in a purposeful, meaningful, and inquiry-based manner. The primary aims of Unit 16•2 are helping readers: (a) construct an explicit understanding of functions, (b) recognize the importance of functions as a way of integrating many aspects of mathematics including algebra, and (c) see how the investigative approach can be useful in exploring this topic with elementary-level children.

**Investigation 16.5: In-Out (Function) Machines** (pages 16-20 to 16-22 of the Student Guide) was designed to help accomplish all three aims just outlined. **Part I** illustrates how in-out machines can provide an informal and relatively "concrete" model of functions, one that can be used as early as kindergarten. **Part II** illustrates the next step in learning about functions: summarizing verbal or written arithmetic rules as algebraic expressions. **Part III** demonstrates how students can be guided to discover the critical attributes of a function and explicitly define the concept as a type of relationship in which each input is associated with one and only one output. Note that this investigation also serves to ease the introduction of functions to adult students by helping them build an intuitive understanding of the concept from their existing knowledge, describing functions explicitly in terms of algebra, and finally defining the concept clearly and precisely. (See pages 465 and 486 for further discussion of this reader inquiry.)

**Investigation 16.6: Arrow Diagrams** (page 16-23) illustrates another way of representing relationships, in general, and functions, in particular. It can also underscore a general definition of functions as a special kind of relationship in which each \( x \) value has one and only one \( y \) value. Although some mathematics educators might think arrow diagrams are too abstract and confusing for adult or intermediate-level students, our experience is that this relatively concrete representation can be helpful in explicitly defining functions.

**Investigation 16.7: Bridges** (pages 16-25 and 16-26 of the Student Guide) illustrates another way in which to introduce functions informally to children. It further illustrates how algebra can be used as a shorthand for summarizing the rule of a functional relationship.

**Probe 16.3: Linking Different Representations of Functions** (pages 16-27 and 16-28 of the Student Guide) illustrates various ways functions can be represented. This probe may be particularly important for many adult students to do because, as products of the traditional skills approach, they may not recognize how formulas, graphs, and real data are interrelated. For example, some may know that the general equation \( y = mx + b \) is represented by a straight line and that \( m = \text{slope} \) and \( b = \text{the y-intercept} \). However, they may have no idea how such an equation, each of the terms in the equation, or its graph relate to real-world situations (e.g., in Part II, that the \( y \)-intercept represents the initial amount of the debt).
SAMPLE LESSON PLANS

Project-Based Approach

Using the SUGGESTED ACTIVITIES on pages 477 to 479 of this guide as a menu, an instructor could have small groups of students choose a project. Note that choices 1 to 3 and 5 highlight uses of algebra or functions. Choices 2 to 9 involve developing instructional materials or lessons on these topics. Choices 4 to 11 entail actually teaching or assessing children. Choices 11 to 14 require the critical analysis and evaluation of instructional materials. Choices 2, 3, 8, 12, and 13 involve technology in one way or another.

Another possibility is to have students actually carry out a project such as that described in Problem 2: Temperature Conversions or Problem 8: Car-Resale Value of Investigation 16.1 (pages 16-4 and 16-6 of the Student Guide). Such a project would purposefully involve them in (a) collecting, analyzing, and describing real data, (b) using computers to access and navigate the world wide web, (c) mathematical inquiry to find a way of translating Fahrenheit readings to Celsius readings, and (d) using algebra as a pattern generalizer. Discussing their experience could provide a basis for highlighting many of the teaching points in chapter 16.

Single-Activity Approach

A class could spend one or more periods on Investigation 16.1: Some Problems to Consider (pages 16-4 to 16-6 of the Student Guide) alone. Investigation 16.B: Guided Exploration of an Algebra Problem (on pages 469 to 471 of this guide) can, likewise, take a full class period or more and provide a broad overview of the topic of algebra and functions. Investigation 16.C: Exposed Surfaces (page 472 of this guide) or Investigation 16.D: The Period of a Pendulum (pages 473 and 474 of this guide) can serve to illustrate how a science lesson could provide a basis for instruction on algebra and functions.

Multiple-Activities Approach

Lesson 1. The illustrative lesson outlined below is based on reader inquiries in the Student Guide and can provide relatively broad coverage of the topic algebra.

1. Tackling and discussing selected problems of Investigation 16.1: Some Problems to Consider (pages 16-4 to 16-6) can serve to underscore two key roles of algebra and two related teaching points: (a) Algebra serves as a shorthand for describing patterns and relationships, and algebra instruction should build on children’s efforts to induce and describe arithmetic and geometric patterns and relationships. (b) Algebra can serve to make problem solving more efficient or even possible, and algebra instruction should grow out of attempts to solve worthwhile problems.

Problem 2 provides a particularly good opportunity for accomplishing the first set of aims outlined above. Typically, almost no adult students can recall the formula for converting a Fahrenheit reading to a Celsius reading. An instructor can read the first paragraph of Problem 2 as background and ask, "What if we had to convert 50˚F to C˚? What do we already know about these two temperature scales? Usually students will know the freezing and boiling points of water in each scale. After listing these readings on the board an instructor can insert 50˚F as shown below:

\[
\begin{align*}
\text{freezing point of water} & \quad 32^\circ \quad 0^\circ \\
\text{boiling point of water} & \quad 212^\circ \quad 100^\circ 
\end{align*}
\]

Asking adult students to use this information and their arithmetic knowledge can often result in the same conjectures and give and take outlined in Questions 2 to 11 obtained with intermediate-level students. For example, often an adult student will propose setting up the proportion \(\frac{32}{50} = \frac{0}{C}\). When this doesn’t work, someone frequently suggests the proportion \(\frac{50}{212} = \frac{C}{100}\). After students determine this results in an answer of about 23.6˚C, an instructor can ask if it makes sense. Often someone will note that, whereas 50˚F is about a tenth of the way from 32˚ to 212˚F, 23.6 is about one-fourth of the way from 0˚ to 100˚C. With some reflection, students should realize they need to take into account that the Celsius scale is more compressed and that 50˚F is 18 units from the common base of comparison (the freezing point of water), not 50:

\[
\begin{align*}
\text{F} & \quad \text{C} \\
\text{diff} = 180 & \Rightarrow 32 \quad 0 \quad 50 \quad C \quad 212 \quad 100 \\
\text{diff} = 100 & \\
\end{align*}
\]
So the proportion should be:

\[
\frac{50 - 32}{212 - 32} = \frac{C^\circ}{100}
\]

\[
C^\circ = \frac{100(50 - 32)}{180} = \frac{5}{9} (50 - 32) = \frac{5}{9} (18) = 10^\circ C
\]

After converting 77°F and 95°F by using the arithmetic procedure above, a class can be asked to consider how any Fahrenheit reading F can be converted to an equivalent Celsius reading C. Students should notice by now that subtracting 32 from the Fahrenheit reading and multiplying this difference by \(\frac{5}{9}\) is always repeated. So the general formula is \(^\circ C = \frac{F - 32}{9} \times \frac{5}{9} = 10^\circ C\).

Solving and discussing Problem 8, for example, can illustrate how algebra can be useful in solving problems. It could also serve to connect algebraic equations to graphing and to introduce the technology of graphing calculators.

2. Completing and discussing Question 1 of Probe 16.1: Uses of Letters (page 16-11) can provide a basis for explicitly identifying and distinguishing among the four types of variables. Most students have little difficulty with these distinctions, except for cases where more than one use may be applicable. For example, given \(c = \pi d\), where \(d = 2\), the variable \(c\) serves as a specific unknown; \(c = \pi d\) can also serve as a pattern generalizer (algebraically represents the relationship between the circumference and diameter of a circle). Furthermore, the variables \(c\) and \(d\) also meet the definition of a varying value: the value of one can depend on the other. In other words, the type of variable a letter represents depends on the situation. (Note that this is point of Question 2.2)

3. Completing and discussing Part I of Probe 16.2: Algebraic Symbol Sense (pages 16-13 and 16-14) can serve to illustrate a common difficulty with representing mathematical processes. Typically, at least a few students incorrectly translate "six times as many students as professors" into \(6s = 6 \pi\), whereas others correctly translate it into \(5 = \pi \theta\). Likewise, at least a few may incorrectly translate "Cecil made $50 more than Ross" into \(C + 50 = R\). An instructor can then model the investigative approach by asking students how they can prove one or the other correct. Someone may suggest, for instance, the informal strategy of substituting numbers. If no one suggests it, an instructor can ask the class how a balance-beam analogy might help children in correctly translating the problem situations into algebraic equations.

Discussing Questions 1 and 2 of Part II of Probe 16.2 can serve to underscore the point that students should be encouraged to draw on their existing knowledge to make sense of algebraic notation and manipulations. For example, is \(2l = l^2\)? By considering the arithmetic meaning of each term, students should recognize that \(2l\) (\(l\) doubled) is equivalent to \(l + l\) and that \(l^2\) (\(l\) squared) equals \(l \cdot l\). Substituting numbers for \(l\) should quickly convince those who already do not realize that \(l + l\) almost never equals \(l \cdot l\). Competing the Questions for Reflection can illustrate how Algebra Tiles can be helpful in making such connections more concrete. For example, \(2l\) could be represented by two short blue rods (see page 16-14), whereas \(l^2\) would be represented by a small blue square (again see page 16-14).

4. Discussing Problem 1 and the Questions for Reflection of Investigation 16.4: Using Algebra Tiles to Solve Algebra Problems & Equations Concretely (page 16-18 of the Student Guide) can illustrate the usefulness of this manipulative and an area model in understanding the multiplication and factoring of polynomials and in rediscovering FOIL. One of the biggest problems adult students have with these activities is using the tiles to fill in the area of a rectangle. For example, for \((x + 3)(y + 2)\), some do not realize that the area representing the partial product \(x \cdot y\) must be \(x\) long and \(y\) wide (represented by the medium-size blue rectangle shown on page 16-14 of the Student Guide). Instead, they try to fill the space with, for example, two small blue \((x \times y \text{ or } x^2)\) squares. Moreover, when modeling the factoring of polynomials such as \(2x^2 + 8x + 6\), some students have no difficulty selecting the correct tiles but have difficulty arranging them to make a rectangle. In both cases, discourage such students from using "math magic" (e.g., FOIL) to first determine their answer and then figure out how to model the expression. Instead, when multiplying polynomials, encourage them to think in terms of the area model and, as suggested above, what the partial products should look like. When factoring polynomials, encourage them to experiment with different arrangements until the find one that forms a rectangle. After all, this is how children will have to use the tiles.

Lesson 2. For a relatively comprehensive overview of functions, an instructor might build a
lesson around the following portions of reader inquiries in the Student Guide:

1. Completing and discussing one or two examples from Part I of Investigation 16.5: In-Out (Function) Machines (pages 16-20 and 16-21) should be sufficient for the vast majority of adult students to understand the model and appreciate its value as an informal analogy for functions. Completing and discussing Part II (page 16-21) can help illustrate the value of using algebra as a shorthand for representing a rule. Note that while Example K is relatively easy for most students, Example L is relatively difficult. Completing and discussing each of the questions in Part III (page 16-22) can help students construct an explicit understanding of the defining characteristics of a function. Almost always, some students incorrectly choose a (an input must produce a unique output) for Question 3. They can be prompted to reconsider their answer by asking the class, "Any counterexamples for this conjecture?" Often someone will offer the counterexample of an in-out machine with the rule multiply by zero or subtract the input by itself (Example O). In either case, all the outputs are zero and hardly unique. Example T in Question 4 is particularly puzzling for some. Because it involves a two-part rule (If the input is even, add four; if it is odd, add three), they conclude that T is not an example of a function. Such students can be prompted to consider Example T in terms of their previously derived definition: Does each input have one and only one output?

2. Completing Part II of Investigation 16.6: Arrow Diagrams (page 16-23) can help cement students' understanding of functions. Reviewing the concept of functions with this medium underscores visually that it involves a relationship where a value of one variable \( x \) is associated with one and only one value of another variable \( y \). Although Item B and D might still mislead a few students, a review and discussion of the reasons for the answers to these items often helps.

3. Investigation 16.7: Bridges (pages 16-25 and 16-26) can serve to illustrate how functions can be explored informally and summarized symbolically as algebraic expressions. Particularly for students having difficulty with the latter, working with actual Cuisenaire rods and in small groups can be most helpful.

4. Completing and discussing Part I of Problem 6.3: Linking Different Representations of Functions (page 16-27) can help students see the connections among sets of ordered pairs, data tables, algebraic equations, and graphs. Some students may have difficulty visualizing sets of ordered pairs, data tables, or algebraic equations as graphs. In the first two cases, they can be encouraged to graph the data given, in the third case, they can be encouraged to substitute various values for \( x \), determine the corresponding value of \( y \), and graph the points. This can provide an opportunity for underscoring that a graphing sense is not something that happens quickly but evolves with experience and knowledge.

An instructor can use Question 3 in Part I as an opportunity to further underscore the defining characteristic of a function (i.e., a relationship in which each \( x \) value has one and only one \( y \) value). This question can also serve as a basis for discussing the underlying rationale for the "vertical-pencil" rule for identifying which graphs represent a function, a rule that is often taught without explanation and simply learned by rote.

Part II of Probe 16.3 (page 16-28) can help students relate (linear) graphs to a real-world situation. Specifically, as Part IV of Investigation 12.3 (pages 12-11 and 12-12) illustrated earlier, the \( y \)-intercept can represent the starting amount (initial debt); the slope, the regular payments; and the \( x \)-intercept, the anticipated time when the initial debt is paid off. Discussing the Questions for Reflection can further help many students start building a better graph sense.

SAMPLE HOMEWORK ASSIGNMENTS

Lesson 1

Read: Unit 16 1 in chapter 16 of the Student Guide.

Study Group:

- Questions to Check Understanding: 1a to 1i, 5, 6b, 6d, 6e, 7c, and 8b (pages 479 and 480).
- Writing or Journal Assignments: 2 and 3 (page 481).
- Problem: Patio Expansion (page 482).
- Bonus Problem: Algebraic Proof for Jason's Algorithm or Algebraic Proof for a Finger Method for Multiplying (page 483).
Individual Journals: *Writing or Journal Assignment* 1 (page 481).

**Lesson 2**

Read: Unit 16•2 in chapter 16.

**Study Group:**

- *Questions to Check Understanding*: 1j to 1l and 1 (pages 479 and 480).
- *Writing or Journal Assignment*: 7 (page 482).
- *Bonus Problem*: Generalizing a Cube Pattern (page 483).

**FOR FURTHER EXPLORATION**

**ADDITIONAL READER INQUIRIES**

**Probe 16.A** (page 467)

Choosing Instructional Ideas to Promote the Symbolization of Method can help underscore that there are many worthwhile tasks involving prealgebra or algebra suitable for elementary instruction. In fact, essentially any task that encourages children to focus or reflect on a solution method is helping to set the stage for algebra.

**Investigation 16.A** (page 468) and
**Investigation 16.B** (pages 469 to 471)

Building on Arithmetic and Geometry Experiences and Guided Exploration of an Algebra Problem both illustrate problem-solving based activities that involve symbolizing mathematical processes. More specifically, they exemplify the teaching point that using letters as a shorthand for representing arithmetic and geometric processes should grow out of meaningful and purposeful activities—efforts to find and to describe arithmetic or geometric patterns or relationships.

**Investigation 16.C** (page 472) and
**Investigation 16.D** (pages 473 and 474)

Exposed Surfaces and The Period of a Pendulum both illustrate how a science lesson can provide a basis for informally introducing functions in a purposeful and interesting manner.

**Investigation 16.E** (pages 475 and 476)

Connecting Different Representations of Functions is an extension of Probe 16.3: Linking Different Representations of Functions (pages 16-27 and 16-28 of the Student Guide).

**QUESTIONS TO CONSIDER**

1. During student teaching, Missy tried to have her seventh-grade students discover the rules of order. She gave the class the following verbal expression to solve: The fifth root of 729 times the sum of two cubed plus two to the zero power raised to the negative fourth power divided by the inverse of 54. As the lesson evolved, Missy never had the opportunity to help her students discover the rules of order. What is wrong with the example Missy chose as the basis of her lesson?

2. A columnist observed that the Democratic presidential candidate “Clinton is the obvious choice [over the Republican candidate Bush] for the majority of those seeking a change of leadership and policy . . . . But Perot [an independent third-party candidate] throws an X factor into the equation.”† How is the letter X used in this context (i.e., what kind of variable is it)? Briefly justify your answer by indicating what the columnist meant by the quote above.

3. Fifth-graders were given the following word problem: If three and twice a number are added, the result is 26. One group started by estimating what number doubled would be about 26. They agreed that the number had to be less than 12 (because 12 doubled is 24 and 3 more is 27) but more than 11 (because 11 doubled is 22 and 3 more is 25). They finally settled on an answer of 11.5. (a) These children used what type of arithmetic solution strategy to solve for the unknown number—a forward process or an undoing process (see Box 16.1 at the top of page 477)? (b) Is their arithmetic strategy consistent or inconsistent with an algebraic solution strategy? Briefly explain why or why not. (Text continued on page 477.)

† David Broder of the Washington Post made this observation during 1992 presidential contest (*Newsweek*, 10/14/92).
Probe 16.A: Choosing Instructional Ideas to Promote the Symbolization of Method

Using letters as a shorthand for representing mathematical processes (the symbolization of method) requires reflection on how problems are solved. This probe asks you to consider what elementary-level activities focus on process and, thus, provide a basis for an ability to symbolize methods algebraically. Discuss your conclusions with your group or class.

1. Circle the letter of any of the following activities that focus on process (not on the answer). Briefly justify each answer.
   
a. Encouraging students to invent their own written or mental procedure for adding multidigit numbers.

b. Analyzing the inputs and outputs of an In-Out Machine to determine the machine’s internal rule (see, e.g., Investigation 16.5 on pages 2-20 to 2-22 of the Student Guide).

c. Analyzing sets of data such as that below to play Guess My Rule (see page 2-25 of the Student Guide).

| □  | 4 | 5 | 6 | 7 | 8 | 9 |
| ▲  | 5 | 7 | 7 | 9 | 9 | 11 |

What is the rule for changing a □ into ▲?

d. Memorizing the basic multiplication facts by rote.

e. Asking children to share their thinking strategies for reasoning out basic multiplication combinations.

f. Practicing a multidigit renaming procedure for subtraction by doing worksheets.

g. Asking students to justify their conjecture and analyzing their reasoning.

h. Asking children to solve strategy problems such as the problem below:

   □ What Calculator Keys? (♦ 3-5). Miguel had to determine the distance around a square whose side was 9.5 meters. What should Miguel key into his calculator to determine the answer?

i. Encouraging children to discuss and to evaluate what strategy would be the most effective for estimating the area of a circle.

j. Completing a worksheet on rounding two-digit numbers to the tens place, rounding down if the ones digit was 0 to 4 or rounding up if the ones digit was 5 to 9.

2. (a) Would explaining to children that missing-change change-add-to problems can be solved by subtracting provide a basis for the symbolization of method—that is, encourage a focus on process (not answers)? (b) Would it help them consider forward operations as an alternative to undoing operations? Why or why not?

3. Would encouraging children to solve a missing-addend equation such as $5 + □ = 9$ by estimating the value of the unknown, adding it to 5, comparing the sum to 9, and adjusting the value of the unknown involve a forward or an undoing operation?
Investigation 16.A: Building on Arithmetic and Geometry Experiences

- Algebra + geometry (area) + multidigit multiplication
- 4-8 - Groups of four + class discussion

Algebra should be introduced as a shorthand for summarizing students’ arithmetic or geometric discoveries. To see how this can be done, try the following sample investigation yourself, either on your own or with your group. Share your results with your group or class.

Part I: A Problem (4-8)

- **Patchwork Napkins.** Eugene and Emma sewed together 1-inch by 1-inch squares of scrap material to make a square napkin 10 inches long on each side. The outer border of the napkin consisted of red squares and the center of the napkin consisted of assorted browns. Their mother so liked the napkin, she asked the children to make a set of 6. For each napkin, what is the total number of squares, the number of red squares, and the number of brown center squares the children would need?

1. Presented with the problem above, Magnus retrieved two sets of plastic 1-inch by 1-inch square tiles. But after considering the problem further, remarked, "Boy, this is going to take a lot of blocks." Magnus’s comment provides an opportunity to underscore what lesson(s) about mathematics?

2. A class of fourth- and fifth-grade students quickly determined the total number of squares needed but disagreed about the number of red border squares needed. Whitney noted, "Ten on each side, so 40 altogether." Antoine suggested 36. (a) Who, if either, is correct? Why? (b) What problem-solving heuristic would help resolve this disagreement?

3. Miguel suggested that the top and bottom rows of the border each had 10 red squares and the left and right columns of the border had 8 red squares each. Akiko noted that the border consisted of four sides of 8 squares plus 4 corner squares. What other ways could the border squares be counted?

4. After determining the total number of squares and the number of red border squares, how could the number of brown center squares be determined?

Part II: Extensions (4-8)

- **Patchwork Placemats.** Eugene and Emma’s mom so liked the napkins, she asked them to make placemats. The children decided on square placemats of the same design 15 inches on each side. For each placemat, what is the number of red border squares, and the number of brown center squares?

- **Patchwork Tablecloth.** Mom next commissioned Eugene and Emma to make a patchwork tablecloth for their dinette table. If the same pattern was used to make a square tablecloth 60 inches on the side, what is the total number of squares, the number of red border squares, and the number of brown center squares needed?

Part III: A General Plan (4-8)

- **Patchwork Doilies.** Eugene and Emma’s mom had become obsessed with their patchwork handicraft and now wanted square doilies of all sizes. Using algebra as a shorthand, summarize how Eugene and Emma could, for a square doily of any given length, determine (a) the total number squares and (b) the number of red border squares. (Let \( L \) = the length of any given doily.) (c) If \( T \) = the total number of squares needed, \( B \) = red border squares needed, and \( C \) = brown center squares needed, write an algebraic number sentence for determining the number of brown center squares needed if the total and the number of red squares is known.

Part IV: Simplifying Expressions (6-8)

Before trying to write an algebraic equation for determining the number of brown center squares in terms of \( L \) (the length of any given doily), it may help to simplify the algebraic equation for \( B \) written for Part III. One group of children devised the following formula for \( B \): \( B = (L - 2) \times 4 + 4 \).

How could this expression be made less cumbersome? How could a teacher help elementary-level children interpret \((L - 2) \times 4\) in terms of arithmetic knowledge so that they could begin the process of simplifying the expression?

Part V: Extending Algebraic Explorations (6-8)

- **Patchwork Crafts Revisited.** What if Eugene and Emma’s mom wanted rectangular doilies, tablecloths, napkins, and so forth? Write an algebraic equation that could be used to determine the number of red border squares that would be needed for a patchwork craft of any given length and any given width. Do the same for determining the number of brown center squares.

† Based on Introductory Algebra (Chapter 2) of *A Collection of Math Lessons From Grades 6 to 8* by Marilyn Burns and Cathy McLaughlin © 1990 by The Math Solutions Publications.
Investigation 16.B: Guided Exploration of an Algebra Problem†

Investigating patterns and relationships + geometric concepts (perimeter and area) + symbolization of method + concept of variable + graphing

- Small groups of four + class discussion

This investigation illustrates how students can be guided to discover (a) the value of algebra as a shorthand, (b) its usefulness in modeling problems, and (c) the relationships among algebra and other representations. It also entails using variables in several different ways. To see what is involved, work through the investigation yourself on your own or, better yet, with your group. Discuss your results with your group or class.

Part I: A Problem (◆ 3-8)

Consider the following problem:

A Fussy Builder. Mr. Femish wanted to build a patio in his back yard with 1-yard x 1-yard square patio blocks. To hold the blocks in place and keep plants from rooting under his patio, Mr. Femish planned to surround the outside edge of the patio with a weed barrier—a flexible plastic liner secured to the ground with stakes. He had just enough weed barrier to surround 18 yards of patio. Illustrated in Figure A below is Mr. Femish’s partially complete patio. How could he add square patio blocks to it so that the completed patio had a perimeter of 18 yards? Note that an added block must touch along an entire side of an existing block.

Using square tiles or graph-paper drawings can be helpful.

Teaching Tip. To facilitate analyses requested later, have children use one color for the original 6 tiles and a different color for any added tiles (see Figure B below).

Part II: Patterns (◆ 3-8)

Does adding a block (tile) always make the perimeter larger? Does it always make the area larger?

To address these issues, summarize your answers to the questions below in the table at the bottom of this page. To make recording your answers easier, use P as a shorthand for the perimeter (e.g., the algebraic expression \( P + 2 \) can serve as a shorthand for writing out the perimeter would increase by two) and A as a shorthand for area (e.g., the algebraic expression \( A + 1 \) can serve as a shorthand for writing out the area would increase by one).

1. (a) What does adding a square block (tile) that touches another along one entire edge do to the perimeter? Why? (b) What happens to the area?
2. (a) What effect does adding a tile that touches another along two edges have on the perimeter? Why? (b) What happens to the area?
3. (a) Adding a tile that touches other tiles along three edges does what to the perimeter? Why? (b) What happens to the area?
4. (a) How could a new tile be added so that it touches on four sides? (b) What would happen to the perimeter? (c) What issue does the previous question raise? (d) What would happen to the area?

Table for Part II

<table>
<thead>
<tr>
<th>Effect on the perimeter</th>
<th>Effect on the area</th>
</tr>
</thead>
<tbody>
<tr>
<td>New tile touches on 1 side</td>
<td></td>
</tr>
<tr>
<td>New tile touches on 2 sides</td>
<td></td>
</tr>
<tr>
<td>New tile touches on 3 sides</td>
<td></td>
</tr>
<tr>
<td>New tile touches on 4 sides</td>
<td></td>
</tr>
</tbody>
</table>
Part III: Extension (4-8)

As you probably have discovered, there are various ways to add tiles to Figure A on the previous page to make a patio with a perimeter of 18 yards. Now consider the following questions:

1. What is the fewest number of tiles that can be added to the Figure A to obtain a perimeter of 18?

2. (a) What is the largest number of tiles that can be added to the Figure A to obtain a perimeter of 18? (b) What is special about this solution?

3. For rectangles with a perimeter of 18, write an equation to show how this perimeter can be determined from the rectangle’s length and width. Use \( l \) as a shorthand for the length of the rectangle, \( w \) as a shorthand for the width of the rectangle.

4. Devise an equation that shows how the perimeter of any rectangle can be determined from its length and width. If need be, students can be encouraged to use tiles or graph paper drawings to determine the perimeter of various rectangles (e.g., \( l = 3 \) and \( w = 2 \), \( l = 5 \) and \( w = 4 \), \( l = 8 \) and \( w = 5 \), and \( l = 10 \) and \( w = 6 \)).

Part IV: Further Explorations (5-8).

Question 1 below may arise from the work above and can be solved intuitively. Question 2 asks how a Math Balance could be used to answer Question 1. Question 3, which is appropriate for seventh- and eighth-graders, asks how algebra could be used to answer Question 1 in a relatively efficient manner.

1. (a) Use tiles and a separate piece of paper to create a bar graph that summarizes your findings of Part IV. (The lengths can be represented along the horizontal axis; the corresponding widths, along the vertical axis.) (b) Summarize these same data by plotting points in Graph A below:

2. (a) The graph above can be represented by what (algebraic) formula? Hint: Simplify the formula you devised for Question 3 of Part III (e.g., \( 2l + 2w = 18 \)) so that \( w \) stands by itself on one side of the equation. (b) Check the formula you devised for the previous question by

Graph A

(c) Would it make sense to connect the dots in this graph? Why or why not? (d) What would a dot half way between a length of 6 and 7 and half way between a width of 2 and 3 represent, and what would the perimeter of the figure be? (e) What computational skill would be practiced answering Question d?

2. (a) Use tiles and a separate piece of paper to create a bar graph that summarizes your findings of Part IV. (The lengths can be represented along the horizontal axis; the corresponding widths, along the vertical axis.) (b) Summarize these same data by plotting points in Graph A below:

Part V: Exploring the Relationship Between the Length and Width of a Rectangle with a Perimeter of 18 (5-8)

How does width change as length changes if the perimeter of a rectangle is held constant? To address this issue, the first question below asks students to construct a graphic representation of the solutions to the questions in Part IV; the second question, an algebraic representation.

1. (a) Use tiles and a separate piece of paper to create a bar graph that summarizes your findings of Part IV. (The lengths can be represented along the horizontal axis; the corresponding widths, along the vertical axis.) (b) Summarize these same data by plotting points in Graph A below:
entering a value from 1 to 8 for \( l \). Plot the \( l \) value and the resulting \( w \) value on Graph A. The plotted point should cover one of the points already on your graph. (c) Mr. Aberjeen pointed out that the formula the class derived would allow them to figure out a width for any given length of a rectangle with a perimeter of 18. Alexya asked, "Why do I need a formula to do that? I can already figure out the width if I know the length of such a rectangle. All I have to do is look at my chart or my graph." How could Mr. Aberjeen help Alexya to appreciate the value of the formula? (d) Mr. Aberjeen asked Alexya what the width of a rectangle would be if he knew the perimeter was 18 and the length was 5.84. What would the width be if the length were 2.85693?

Part VI: Exploring the Relationship Between the Area and Length of a Rectangle of a Given Perimeter

If the perimeter of a rectangle is held constant, how does area change as the length changes? Question 1 asks students to construct a graphic representation of how area varies with length for a rectangle with a constant perimeter of 18; Question 2, an algebraic representation.

1. In Graph B below, plot the point that indicates the area of a rectangle with a perimeter of 18 when the length is 1. Do the same for lengths of 2 to 8.

Graph B

<table>
<thead>
<tr>
<th>Length</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

2. The results in Graph B can be represented by what (algebraic) formula? Hint: What is the formula for the area of a rectangle? From Part V, what is the known value of \( w \)? How can you use these two pieces of information to devise a formula that involves only area and length?

3. To foster reflection, a teacher can pose such follow-up questions as:
   a. Would it make sense to connect the points?
   b. If the points were connected, what would a position on this line between two such points imply?
   c. How would you interpret the point the line intersects on the horizontal axis?
   d. What length would give the maximum area for a rectangle with a perimeter of 18?
   e. What length would give the maximum area for a rectangle with a perimeter of 16? 12? 9?

**Teaching Tip.** Note that Part VI could also be introduced with an applied problem such as:

- **Designing Mobile Home.** For a fixed perimeter, does a rectangular mobile home make the best use of the available space—that is, does it maximize the area?

Questions for Reflection

1. Amnon knew the perimeter of his original figure (see Figure A on page 469) was 12. He added a dark block as shown in Figure B (also on page 469). Instead of recounting to determine the perimeter, what intuitive shortcut could he use to determine the new perimeter—a shortcut that could logically lead to the algebraic short cut \( P + 2? \)

2. Troy interpreted the equation \( p = (l \times 2) + (w \times 2) \) as, "The perimeter can be determined by doubling the length and doubling the width and adding the two together." Wade argued that this is not what the equation said. Who is correct? Why? How is the incorrect child’s knowledge incomplete?
Investigation 16.C: Exposed Surfaces

Functions and surface area  7-8  Small groups of about four

This investigation illustrates a science-related application of functions and involves exploring the mathematical topic of surface area. To see what is involved, try it with your group. Discuss your findings with your class.

Problem 1: Is Heat Loss a Function of Shape? In cold climates such as the arctic region, the greater the surface area of an animal, the greater the heat loss. Thus, natural selection favors shapes that minimize surface area and heat loss. Assume that an animal is 8 blocks in size (volume). What shape would expose the least surface area? Include the following shapes: snake-like (1 x 1 x 8), thin prism (1-block high, 2-blocks wide, and 4-blocks long), and a cube (2 x 2 x 2). Children may also want to experiment with other types of shapes such as those below. Determine the surface area of each shape. Summarize your data in Table A to the right. What shape has the greatest surface area and, hence, the greatest heat loss? What shape has the least surface area and, thus, the least heat loss?

<table>
<thead>
<tr>
<th>Shape</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A

Table B

Problem 2: Is Heat Loss a Function of Size? Consider one shape—the cube. (a) As cubes get larger does the ratio of exposed surface area to volume grow, decrease, or stay the same? (Hint: Consider how many blocks are needed to construct 2 x 2 x 2, 3 x 3 x 3, 4 x 4 x 4, and 5 x 5 x 5 cubes.) Summarize your data in Table B below. (b) What are the implications of your results? That is, would heat loss increase, decrease, or remain constant as the volume of a cube increases? Would a cube-shaped animal be better off small or large or would it not matter?

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (number of blocks long)</td>
<td>Volume (number of blocks)</td>
<td>Surface area</td>
<td>Ratio of C to B</td>
</tr>
</tbody>
</table>
Investigation 16.D: The Period of a Pendulum

- Collecting and analyzing the data of a controlled experiment
- Small groups of students

This integrated science-mathematics activity illustrates another science-related application of functions. It addresses the following question: What determines how fast a pendulum swings and, thus, its period (the number of swings per unit of time)? With your group or class, try the experiment yourself to see what is involved.

Show the class a pendulum and ask them what determines how fast it swings. Intuitively, it would seem that weight would be an important factor. Are there other factors that might be important? Solicit suggestions. What factor or factors do you think affect how fast a pendulum swings?

In addition to weight, some factors that students might suggest include (a) the height from which the pendulum is released, (b) how hard the pendulum is first pushed, and (c) the length of the pendulum. Note that in a controlled experiment, scientists change only one factor and keep all other factors constant. Why is only one factor at a time varied?

Before conducting the experiment, have the students predict what factor or factors they believe will have an effect. Individually or as a group have them fill out the Prediction Chart in the bottom left corner of the page. Compile the predictions for the whole class.

Have each group set up a pendulum using string, a paper clip, and a ball of clay. Secure the string to an eyehook, ring stand, or other apparatus that allows the pendulum to swing freely. (Note that friction will slow the pendulum gradually and introduce some error into your measurements.) Secure the paperclip to the other end of the string and use it to hook the clay ball.

Part I: Weight

Test the effect of weight on the period of a pendulum. For example, have each group weigh out 10 grams of clay using a balance beam or a weighing device. Fix the length of the string at 60 cm. Start each swing with the clay at a height of 30 cm. above the table. To control the force with which it is propelled, have a student let the pendulum slip out of his or her fingers (no initial push). Have the students count the number of complete swings (times back and forth) in a set time frame (e.g., 30 seconds). One student in the group can serve as the timer, and the others can serve as counters. Record the number of swings in Table 1 on the next page.

To increase the accuracy of measurements and to provide a purposeful context for considering averages, have students repeat the experiment three times with each weight. As a class, consider whether to use a mean, median, or mode.

Repeat the same procedure with clay balls weighing 20 grams, 30 grams, and 40 grams. Record the data of these trials. Does the number of swings change appreciably as weight is increased? Note that this question raises the following issues: What is appreciable? Is a difference of $\frac{1}{10}$ or 1 swing significant? (Probably not.) What might introduce error into the experiment? What did you learn about measurement that is relevant to this situation?
Investigation 16.D continued

**Table 1**

<table>
<thead>
<tr>
<th>Number of Swings in _______ seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
</tr>
<tr>
<td>10 grams</td>
</tr>
<tr>
<td>20 grams</td>
</tr>
<tr>
<td>30 grams</td>
</tr>
<tr>
<td>40 grams</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>Number of Swings in _______ seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Force</td>
</tr>
<tr>
<td>No push</td>
</tr>
<tr>
<td>Push</td>
</tr>
</tbody>
</table>

**Part II: Height**

Using the 40-gram ball of clay, the same length of string, and no initial push each time, repeat the experiment, varying the height from which the pendulum is released on each of four trials. (Note that the datum for a height of 30 cm has already been collected and recorded in Table 1. Rerecord this datum in Table 2.) Does the number of swings change appreciably as height changes?

**Table 2**

<table>
<thead>
<tr>
<th>Number of Swings in _______ seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
</tr>
<tr>
<td>10 cm</td>
</tr>
<tr>
<td>20 cm</td>
</tr>
<tr>
<td>30 cm</td>
</tr>
<tr>
<td>40 cm</td>
</tr>
</tbody>
</table>

**Part III: Initial Force**

Using the 40-gram ball of clay, the same length of string, and a release height of 30 cm, repeat the experiment, but this time, vary the initial force. (Note that the data for no initial push has already been collected and recorded in Tables 1 and 2. Rerecord these data in Table 3.) On the push trial, push the pendulum as hard as possible without disrupting its swing. If the swing is disrupted, try again using a lighter push. Does the number of swings change appreciably when the initial push is changed?

**Table 4**

<table>
<thead>
<tr>
<th>Number of Swings in _______ seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
</tr>
<tr>
<td>20 cm</td>
</tr>
<tr>
<td>30 cm</td>
</tr>
<tr>
<td>40 cm</td>
</tr>
<tr>
<td>60 cm</td>
</tr>
</tbody>
</table>

**Analysis and Conclusions**

1. (a) Graph the data for each investigation. (b) Which factor is most strongly correlated (related) to the pace of a pendulum's swing? (c) What are your conclusions about what affects the period of a pendulum?

2. Conduct additional trials of the factor or factors affecting the number of swings. Graph these data. Describe the functional relationship(s). Indicate whether or not a relationship is proportional. If so, indicate whether it is directly or inversely proportional.
Investigation 16.E: Connecting Different Representations of Functions

- Using two letters to represent functions (varying values) + graphing (intercept and slope) + creating and analyzing tables of data - 4-8 - Individually or small groups of four + class discussion

The aim of this investigation is to help students see the connections among data tables (an arithmetic representation), formulas (an algebraic representation), and graphs (a geometric representation). It can be successfully used as early as fourth grade with students who have had extensive experience playing Guess My Rule (see Probe 2.3 on page 2-25 of the Student Guide). To see what is involved and to perhaps expand your own understanding of functions, try the investigation yourself, either on your own or with your group. Discuss your findings with your group or class.

Part I: Translating Data Tables into Formulas and Graphs

The first activities begin with translating a familiar arithmetic representation of a functional situation into less familiar algebraic and geometric representations.

**Problem 1: Water Bills.** The Auburn Aqua Association (AAA) charged its customers a 1¢ water-use fee for each gallon of water used. It also charged a 1¢ sewage-fee for each gallon of water used. Moreover, AAA charged a flat basic fee of 5¢ a week for simply being hooked up to their water system. On the first day of a week, Mr. Henry went away for vacation but a dripping faucet used 1 gallon of water a day. The unsuspecting vacationer was charged 1¢ for the water and 1¢ for sewage, plus the flat basic fee of 5¢. The next day, Mr. Henry’s charges totaled 2¢ for water, 2¢ for sewage, and 5¢ for the basic fee. These data are summarized in the Table 1 below.

1. (a) What would AAA’s weekly bill total on the third day of Mr. Henry’s vacation? (b) On the seventh day? (c) What would AAA’s weekly bill have been if 0 gallons of water were used (if Mr. Henry didn’t have a leak)?

2. Let \( \square \) represent the number of gallons used and \( \Delta \) represent the total daily AAA charge. Write an equation that would tell you the total daily AAA charge for any given number of gallons used.

3. (a) Complete Table 1 to the right and summarize these data in Graph 1. (b) When \( \square = 0 \), what is \( \Delta \) (the \( \Delta \)-intercept)? (c) In Table 1, what pattern among the \( \square \) entries do you notice? (d) What pattern among the \( \Delta \) entries do you notice? (e) How are these two patterns reflected in the graph?

**Extension for Problem 1.** A leak of how many gallons would result in a daily charge of $5.71 by AAA?

**Problem 2: A Food Plan.** For a monthly fee of $4. Erna could purchase her food at a discount. Table 2 on page 476 compares monthly (nondiscounted) food costs \( \square \) with total monthly food plan costs: fee + discounted food costs \( \Delta \).

1. (a) What rule was used to determine the total monthly cost \( \Delta \)? (b) Complete Table 2. (c) The food plan discounts the cost of food \( \square \) by how much (by what percent)?

Investigation 16.E continued

2. Using $\Box$ and $\Delta$, write an equation that would tell you the total monthly cost of the food plan for any given amount of food purchased.

3. (a) Plot the Table 2 data onto Graph 2 to the right. (b) When $\Box = 0$, what is $\Delta$ (the $\Delta$-intercept)? (c) In Table 2, what pattern among the $\Box$ entries do you notice? (d) What pattern among the $\Delta$ entries do you notice? (e) How are these two patterns reflected in the graph?

+ Extensions for Problem 2. (a) At what point would Erna break even—that is, what monthly food expense $\Box$ would be equal to the total monthly cost $\Delta$? (b) How much would Erna have to spend on food in a month to save $10 with the food plan? (c) Illustrate the various ways can you solve the problems posed in (a) and (b)?

Part II: Translating Formulas into Data Tables and Graphs

For each formula below, create a table of data and plot the data on a graph.

$$(3 \times \Box) + 2 = \Delta$$

Table 3

<table>
<thead>
<tr>
<th>$\Box$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Graph 3

$$(\frac{3}{2} \times \Box) + 2 = \Delta$$

Table 4

<table>
<thead>
<tr>
<th>$\Box$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Graph 4

Part III: Translating Graphs into Data Tables and Formulas

For each graph below, summarize the data in a table and write an algebraic equation.

Graph 5

Table 5

<table>
<thead>
<tr>
<th>$\Box$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Graph 6

Table 6

<table>
<thead>
<tr>
<th>$\Box$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Box 16.1: Forward and Undoing Operations

Consider the problem below:

**Mystery Number** (♣ 4-6). I'm thinking of a number. If I multiply it by 5 and then subtract 2, I get 40. What is the number?

One way of solving this problem arithmetically is to use forward operations—operations implied by the problem. This would involve the following try-and-adjust thinking: What times five is about 40? Eight times five is 40, but when I subtract the two, I get 38. So eight is too small. Let me try nine times five. That's 45; minus two is 43; nine is too big. So the number must be between eight and nine. Let me try 8.5 . . .

Another way of solving the problem arithmetically is to use undoing operations—working backwards (Usiskin, 1988). This would entail adding two to 40, which undoes the subtraction by two specified in the problem, and then dividing by five, which undoes the multiplying by five mentioned in the problem.

4. What do examples A, B, and C below illustrate about functions?

5. (a) What is the rule for In-Out Machine D below? (b) Are any other patterns or rules possible?

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>In</td>
<td>Out</td>
</tr>
<tr>
<td>Delaware</td>
<td>1</td>
</tr>
<tr>
<td>Penna</td>
<td>2</td>
</tr>
<tr>
<td>New Jersey</td>
<td>3</td>
</tr>
<tr>
<td>Georgia</td>
<td>4</td>
</tr>
<tr>
<td>New York</td>
<td>11</td>
</tr>
<tr>
<td>Illinois</td>
<td>21</td>
</tr>
<tr>
<td>Alaska</td>
<td>49</td>
</tr>
<tr>
<td>Hawaii</td>
<td>50</td>
</tr>
</tbody>
</table>

6. Complete the table at the bottom of the page. Note that you may have collected much or all of the data needed when you solved a similar problem—Problem 1 (Painted Cubes) in Investigation 16.1 on page 16-4 of the Student Guide. (a) Which numerical or algebraic expressions involve a constant? (b) Which involve a cubic function? (c) Which involve a linear function? (d) Which involve a quadratic function?

**SUGGESTED ACTIVITIES**

1. Create your own magic trick like that in Problem 4 of Investigation 16.1 (on page 16-5 of the Student Guide). That is, devise your own algorithm. Check out the algorithm with a number of initial choices. Does more than one initial choice produce the same final result? Write a proof for your algorithm using both pictures and algebra.

2. (a) Collect web site, newspaper, or magazine articles that illustrate the everyday use of algebra. (b) Categorize the uses. (c) Describe and illustrate how you could use the material collected for a bulletin board or learning center display appropriate for a grade level of your choice.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of 1x1x1 cubes needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 2 x 2</td>
<td>3 sides 2 sides 1 side 0 sides</td>
</tr>
<tr>
<td>3 x 3 x 3</td>
<td></td>
</tr>
<tr>
<td>4 x 4 x 4</td>
<td></td>
</tr>
<tr>
<td>5 x 5 x 5</td>
<td></td>
</tr>
<tr>
<td>n x n x n</td>
<td></td>
</tr>
</tbody>
</table>
3. (a) Collect web site, newspaper, or magazine articles that illustrate the everyday use of functions. (b) Describe and illustrate how you could use the material collected for a bulletin board or learning center display appropriate for a grade level of your choice.

4. Examine the example of a Math Corner bulletin-board display on page 16-1 of the Student Guide. Solve the problem (Botched Travel Arrangements) illustrated. (b) Find at least five problems that could be used in a similar manner. (c) Present Botched Travel Arrangements to a class of sixth-, seventh- or eighth-graders. What informal and formal solutions did the students use? (d) Present at least one of your problems to a class of intermediate-level students. What informal and formal strategies did they use? (e) Report your results to your class.

5. (a) Devise a lesson in which using algebra would be an integral part. (b) Try out your lesson with a class or group of elementary-level students. (c) Evaluate your lesson and your students' progress. Indicate how you would improve your lesson. (d) Share your results with your class.

6. (a) Find an everyday application of functions and use it as the basis of a lesson plan, including the interpretation of graphs. (b) Try out your lesson with a class or group of elementary-level students. (c) Evaluate your lesson and your students' progress. Indicate how you would improve your lesson. (d) Share your results with your class.

7. (a) Design In-Out Machine activities that would be appropriate for kindergarteners, eighth-graders, and a grade level of your choice. (b) Try out your activities with a class or group of elementary-level students. (c) Evaluate your activities. Indicate how you would improve your activities. (d) Share your results with your class.

8. (a) Devise a lesson that involves one of the reader inquiries in chapter 16 of the Student Guide. (b) Try it out and, if possible, videotape your lesson with a developmentally ready group or class of intermediate-level children. Evaluate your pupils' learning and the strengths and weaknesses of your lesson plan. (c) Share with your class a summary of your lesson plan and the results of your teaching experience.

9. (a) Tutor a group of fourth-, fifth- or sixth-graders for at least three one-hour sessions over the course of at least two months on algebra or functions. (b) Document your tutees' progress. (c) Present your results and conclusions about them to your class.

10. (a) Read Activity File 16.A below, and play Devil's Advocate with a group or class of sixth-, seventh-, or eighth-graders. (b) Videotape or audiotape the lesson and evaluate it. Include in your analysis how the students responded to the game and what they appeared to learn from playing it. Note also how you could improve the game.

Activity File 16.A: Devil's Advocate

Mr. Yant showed his students the expression 3(x + y) and challenged them to state the word phrase correctly. The teacher guaranteed he would write an algebraic expression that correctly interpreted the words the students used. But, whenever a student's words could be interpreted in more than one way, the teacher would deliberately choose to write the expression so that its value was different from 3(x + y). The students were eager to pin down the teacher.

11. (a) Familiarize yourself with one of the instructional resources on pages 16-24, 16-29, and 16-30 of the Student Guide. (b) Use the resource as the basis for teaching a unit to a class of elementary children or tutoring one or more children over the period of a month. (b) Evaluate your students' progress, the value of the instructional materials, and the effectiveness with which you used the materials. Indicate how you would modify the materials or your use of them. (c) Share your results and conclusions with your class.

12. (a) Try out one of the examples of educational software listed on page 16-31 of the Student Guide (e.g., Green Globs or Function Supposes) or an educational program of your own choosing. (b) Evaluate the software in terms of its ease of use and instructional value. (c) Share your evaluation with your class, including a demonstration of particularly interesting or useful aspects of the program review.
13. Visit "The Algebra Word Problem Tutor" web site on percent and probability (http://sands.psy.cmu.edu/ACT/awpt/awpt-home.htm) or a web site of your own choice. (a) Evaluate the web site as a resource for intermediate-level students. Does the web site foster routine or adaptive expertise? (b) Evaluate the value of the web site for a student who is having difficulty representing or solving algebra problems. (c) Share your analyses of the web site with your class.

d. At the elementary level, children should be introduced to variables used as specific unknowns and general unknowns.

e. Children commonly interpret variables, particularly at first, as names for a particular number and traditional elementary instruction did not adequately emphasize variables as general unknowns.

f. Representing a specific unknown could be introduced to third graders by asking them to solve $0 \times x = 0$.

g. The symbolization of method focuses on the solution and requires routine expertise.

h. To foster flexibility, a prealgebra teacher should ask for alternative representations and solutions.

i. Using a try-and-adjust substitution or a working backwards strategy is ineffective in solving algebra problems and should be discouraged.

j. The defining characteristic of an In-Out Machine (a function) is uniqueness (each input produces an output different from that of all other inputs).

k. Graphs A, B, C, and D, but not Graphs E, F, G, and H, represent a function. Hint: The x-axis indicates inputs; the y-axis indicates outputs.

14. (a) Analyze and evaluate a recent algebra textbook used for intermediate-level instruction. Consider to what extent the algebra arises as a natural by-product of students' exploration of arithmetic and geometry, instruction is holistic, and the text helps underscore implicitly and explicitly the value of algebra (the roles it plays). Consider to what extent the text fosters algebra sense and adaptive expertise—that is, builds on students' previous understanding of arithmetic to help them make sense of expressions such as $7x - 5x$ or $2(x + 3)$, or $(4x - 8) \div 2$. Does the textbook make an effort to counter the common difficulty of directly translating word problems into algebraic equations. If so, how? Is any effort made to guide students to discover or rediscover algebraic procedures such as FOIL? Identify whether the suggested instruction best fits the description of a skills, conceptual, investigative, or problem-solving approach. (b) Present your findings and conclusions to your class using appropriate graphs and statistics.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. Circle the letter of any statement that, according to the Student Guide, is true. Change the underlined portion of any false statement to make it true.

   a. Prealgebra (preparations for algebra) should start in grade 5 (the beginning of intermediate-level instruction).

   b. $3x + 2$ is an algebraic equation.

   c. Algebra should arise as a natural by-product of learning mathematical and scientific formulas.
1. The following pairs of $x,y$ coordinates represent a function: $(1,5), (2,10), (3,15), (3,20), (3,25)$. 

2. a. The rule for an In-Out Machine is $10X$, where $X$ is the input (e.g., input of 3 → an output of 30). Is the letter $X$ used as: the name for a number, the representation of a specific unknown, the representation of a general unknown, or the representation of a varying value?

b. The rule for a second In-Out Machine is $10 + Y$, where $Y$ is the input (e.g., input of 3 → an output of 13). Is the letter $Y$ used in the same or a different way from the letter $X$ in Question 2a above? Briefly explain.

c. The rule for a third In-Out Machine is $0Z$ (zero times $Z$), where $Z$ is the input. Is the letter $Z$ used in the same or a different way from the letter $X$ in Question 2a above? Briefly explain.

d. The rule for a fourth In-Out Machine is $Ai$, where $A$ is the input and $i = \sqrt{-1}$. Is the letter $i$ used the same or a different way from the letter $X$ in Question 2a? Briefly explain.

e. In the expression $A - A = 0$, how is the letter $A$ used?

f. In the expression $B^2 - 9 = 0$, how is the letter $B$ used?

3. For each of the following expressions, indicate the role of the variable (letter).

   If a variable is used as the name for a number, write A.
   If a variable is used to represent a specific unknown, write B.
   If a variable is used to represent a general unknown or a pattern generalizer, write C.
   If a variable is used to represent a varying value, write D.

   ____ a. $y = x + 10$
   ____ b. $3\pi = 9.42$
   ____ c. $7 \cdot x = 0$
   ____ d. $7 + x = 10$
   ____ e. $0 ÷ n = 0$

4. Demonstrate or illustrate and explain how a balance-beam analogy could be used to help children understand how to set up an algebraic equation for each of the following: (a) Damir scored three more points than twice the points scored by Festus. (b) Tamika had two fewer points than Asjha.

5. Illustrate and explain how Algebra Tiles could help students comprehend and simplify the following expressions: (a) $3x - 2x$, (b) $4(x + 2)$, (c) $\frac{9x + 6}{3}$. For each case, indicate what meaning or analogy your model is based on.

6. Illustrate how Algebra Tiles could be used to model the following expressions and to determine the product in each case: (a) $(3x + 1)(2x + 1)$, (b) $(3x + 2y)(2x + y)$, and (c) $(3x + 5)(2x + y)$, (d) $(3x + 2y + 4)(x + y + 2)$. (e) The Algebra Tiles models or solutions for parts a to d of this question are based on what meaningful analogy?

7. Illustrate how Algebra Tiles could be used to factor the following expressions:
   (a) $2x^2 + 7x + 6$, (b) $3x + 3xy + 4y + 4$, and (c) $4x^2 + 11x + 4xy + 3y + 6$.

8. Demonstrate or illustrate and explain how a Math Balance could be used to model and to solve: (a) $4 + u = 9$, (b) $3v + 2 = 7$, (c) $2w + 4 = 16$, (d) $x - 4 = 5$, (e) $9 - y = 5$, and (f) $10 - 2z = 4$.

9. Evaluate whether or not each of the following students used the = sign correctly. Briefly justify your answer.

   a. Asked to solve the problem described in Box 16.3 on page 16-32 of the Student Guide, Sharraun wrote:

   $\begin{align*}
   32 + 22a &= 300 \\
   22a &= 300 - 32 - 266 \\
   a &= 268/22 \\
   a &= 12.18 \text{ ft}
   \end{align*}$
b. Given \(2x + (16 ÷ 4) = 20\) and asked to solve for \(x\), Wilfred wrote:

\[
\begin{align*}
16 ÷ 4 &= 4 + 2x = 20 \\
2x &= 16 \\
50x &= 5
\end{align*}
\]

c. To solve a problem, Vladimar wrote:

\[
\begin{align*}
5x &< 9 + 2 \\
5x &< 11 \\
x &= \frac{11}{5} \\
x &= 2.2
\end{align*}
\]

10. a. Set up and solve the following word problem using one variable. Then do the same using two variables.

- Don has three times as many CDs as Bob. Together they have a total of 20 CDs. Find the number of CDs owned by each boy.

b. What common error may some students make in using one or two variables? Illustrate.

c. What can a teacher do to help remedy the common difficulty with writing algebraic equations?

11. a. Which arrow diagrams below represent a function?

- Diagram A above could serve as an alternative representation for which graph(s) on page 16-27 of the Student Guide.

- Georgio's assignment was to draw a graph representing Diagram D. Georgio drew the graph shown below. Is his graph correct or not? If so, explain why. If not, indicate any necessary corrections.

- Diagram E could serve as an alternative representation for which graph(s) on page 16-27 of the Student Guide.

WRITING OR JOURNAL ASSIGNMENTS

1. Evaluate your algebra instruction in light of the guidelines described in this chapter.

2. A new teacher who has been assigned several eighth-grade algebra classes asks your advice about introducing algebra. Describe what advice you would give her for making algebra purposeful, inquiry-based, and meaningful.

3. Miss Brill defined a variable as a letter that stands for the unknown number. Is this an accurate way of describing variable to students? Why or why not?

4. In study hall, a student from someone else’s math class is confused about her algebra assignment. "My teacher," she explains, "said, 'What you do to one side of the equation, you have to do to the other.' I don't understand what he means." Describe how you might help this student understand the rule she described.

5. Write a realistic word problem that involves the following equations: (a) \(3x + 5 = 11\) and (b) \(2y - 10 = 42\).
6. (a) Devise a realistic word problem involving a single variable and represent it as an algebraic equation. Specify what the variable means in real-world terms. (b) Devise a realistic word problem involving two variables and represent it as two algebraic equations. Specify what each variable means in real-world terms.

7. (a) Briefly summarize how Mrs. Stoffel could use her class’ discovery of the constant relationship between the diameter and circumference of a circle (see Box 15.2 on pages 15-23 and 15-24 of the Student Guide) to explicitly make the point that letters (variables) can represent the name of a particular number. Briefly describe how she could use this relationship to explicitly note that variables can represent (b) a specific unknown, (c) a general unknown, and (d) varying values.

8. (a) In each of the following cases, indicate the child’s error. (b) Describe how a teacher could help the student understand why his procedure was wrong.

Case A: In deriving the volume of a cube, Ruben noted that the volume of a prism was \( l \cdot w \cdot h \). As all sides of a cube are equal, he noted, \( s = l = w = h \) and wrote \( V_{cube} = s^3 \cdot 3 = 3s^3 \).

Case B: After developing the formula for converting Fahrenheit readings to Celsius, Colin was given the assignment of using what he knew to develop a formula for converting Celsius readings to Fahrenheit. He wrote:

\[
C = \frac{5}{9} (F - 32)
\]

so \( C + 32 = \frac{5}{9} (F - 32 + 32) \)

and \( \frac{9}{5} (C + 32) = \frac{9}{5} \cdot \frac{5}{9} \cdot F \quad \text{Thus,} \quad F = \frac{9}{5} (C + 32) \)

9. Many students are taught the *vertical-pencil rule* for identifying which graphs are functions and which are not: If a pencil is held vertically (parallel to the vertical or y-axis) and does not touch a graph in more than one place, it represents a function. If the vertical pencil touches at two or more points, then the graph does not represent a function. Unfortunately instruction frequently does not help students understand the rationale for this rule. Based on what you have read in chapter 16, explain or justify this rule.

10. Mr. Yant gave his eighth graders the following problem:

- **Relative Weights** (6-8). Bill weighs three times as much as Don. If Bill’s weight is \( p \) pounds, represent Don’s weight.

Many of the students, particularly those schooled in a “key word” approach, represented Don’s weight as \( 3p \). Asked to explain his answer of \( 3p \), Yamma noted that the problem specified *three times*.

Ashka, who recorded an answer of \( \frac{p}{3} \), tried to explain that Bill weighed more but had difficulty conveying why Don’s weight should be equal to \( p \) divided by 3.

(a) According to the Student Guide how could Mr. Yant help students such as Yamma who translated the problem into the expression \( 3p \)?

(b) How might the problem-solving heuristic of drawing a picture help?

11. Investigation 16.1 on pages 16-4 to 16-16 of the Student Guide illustrated how problems could be used to illustrate the various uses of algebra. Describe how the following problems could be used to illustrate one or more uses of algebra: (a) Botched Travel Arrangements on page 16-1 of the Student Guide, (b) Quark Charges on page 16-9 of the Student Guide, and (c) each of the problems in the Problem section on pages 482 and 483 of this guide.

**PROBLEMS**

- **Patio Expansion** (4-8)

Mr. Bigson’s backyard was triangular in shape and so he started building a patio in the shape shown in Figure A below. After adding another diagonal row of 1 foot x 1 foot square patio blocks, he ran out of blocks (see Figure B below). Mr. Bigson wanted a patio 80 feet in length. How many blocks would he need to buy to complete his patio?
Tower of Hanoi (4-8). The disks on Peg A need to be moved to Peg C. Only one disk at a time may be moved, and a larger disk may not be placed on a smaller disk. What is the fewest number of moves in which this can be done?

Tower of Hanoi Revisited (6-8)

The Tower of Hanoi problem above involves three graduated disks and could be solved using a try-and-adjust strategy. Now consider the same problem with three pegs again but 20 graduated disks. Clearly a try-and-adjust strategy would not be practical. How could you solve the problem now? What is the solution for 20 graduated disks?

A Testy Test-Taker (7-8)

Rodney answered all 30 questions on a test and received a score of 0. "I got some right. How did I get this score?" he protested. If Miss Brill gave 6 points for each correct and -4 points for each incorrect answer, how many questions did Rodney answer correctly?

Algebraic Proof for Jason's Algorithm (7-8)

Pages 212 and 213 of this guide include a description of an algorithm Jason invented to determine the next square from the previous square. Write an algebraic proof to illustrate why his algorithm works.

Algebraic Proof for the Sum of Six Odd Numbers (7-8)

Write an algebraic proof that demonstrates that the sum of six odd numbers must be even. Let a, b, c, d, e, and f represent even numbers.

Super Poison (8 and up)

Recall that in the game Poison (see Problem 10 of Investigation 2.2 on page 2-14 of the Student Guide), players took turns removing one or two items from a row of items and that the goal of the game was to avoid removing the last item. With a row of relatively few items, informal strategies such as that illustrated in Box 2.9 on page 2-32 of the Student Guide work fine. (a) What if the row contained 1000 items though? Devise a winning strategy for this super version of Poison. (b) Devise a winning strategy if the row contained 2000 items. (c) Do the same for 2001 items.

Algebraic Proof for a Finger Method for Multiplying (8 and up)

Box 5.5 on page 5-31 of chapter 5 in the Student Guide illustrated a finger method for determining the products of factors 6 to 10. The solution for 7 x 8 is illustrated below. Why does this method work? Construct a proof. Hint: Multiplication with factors greater than five can be represented as \((5 + x)(5 + y)\). For example, for 7 x 8, \(x = 2\) and \(y = 3\). Thus, 7 x 8 = \((5 + 2)(5 + 3)\). What you need to demonstrate is that \((5 + x)(5 + y)\) equals an algebraic representation of the finger method below. How can this method be illustrated algebraically? Can you show that this expression is equivalent to \((5 + x)(5 + y)\)?

Generalizing a Cube Pattern† (8 and up)

Examine the cube pattern below. (a) How many blocks would be needed to build the next items in this pattern? (b) How many blocks would be needed to build the 100th item? (c) How many blocks would be needed to build the nth item?

† Based on a problem described on page 17 of Algebra for the Twenty-First Century (© 1992 by the National Council of Teachers of Mathematics).
ANSWER KEY for Student Guide

Key for Investigation 16.1 (pages 16-4 to 16-6)

Problem 2. In regard to Question 4, it logically follows that if 212°F = 100°C, then 200°F should be less than 100°C (200 - 32 = 168). However, Alison’s method results in a Celsius equivalent greater than 100°C. Note that Alison was using additive reasoning, not the multiplicative reasoning that underlies proportional relationships.

In regard to Question 11, one way of converting 50°F into °C is set up a proportion.

\[\frac{\Delta}{50 - 32} = \frac{100}{180} = \frac{5}{9}\]
\[\Delta = \frac{5}{9}(50 - 32) = \frac{5}{9} \times 18 = 10\]

By doing several specific examples, students should recognize that a general formula for converting °F into °C can be derived in the following manner:

\[\frac{°C}{°F - 32} = \frac{100}{180} = \frac{5}{9}\]

\[°C = \frac{5}{9} \times (°F - 32)\]

In regard to Question 12, the formula for converting °C into °F could be derived in a similar manner (see Solution A below) or by solving the formula for F (see Solution B below).

Solution A

\[\frac{F - 32}{°C} = 180\]
\[\frac{9}{5}\]
\[F - 32 = \frac{9}{5} \times °C\]
\[F = \frac{9}{5}C + 32\]

Solution B

\[\frac{F - 32}{°C} = \frac{100}{180} = \frac{5}{9}\]
\[\text{If } °C = \frac{5}{9} \times (°F - 32),\]
\[\text{then } \frac{9}{5} °C = °F - 32\]
\[\frac{9}{5} °C + 32 = °F.\]

Problem 6. Algebraically, the problem could be solved by subtracting simultaneous equations:

\[3P + 2S = 1.80\]
\[7P + 5S = 1.45\]
\[P = 0.35\]

Problem 7. With this problem, the need for algebra is clear.

\[3P + 2S = 2.95\]
\[7P + 5S = 7.10\]

Note that subtracting the two equations will not help. To solve for P, the S can be eliminated by multiplying the first equation by 5 and the second by 2. As shown below, subtracting the resulting two equations yields an answer of 0.55.

\[15P + 10S = 14.75\]
\[14P + 10S = 14.20\]
\[P = 0.55\]

Problem 8. Let D = dimes and Q = quarters. If a man has 20 coins consisting of dimes and quarters, then D + Q = 20 and Q = 20 - D. If dimes were worth 25¢ and quarters were worth 10¢, then he would have 90¢ more than if the coins had their normal value. This translates into the equation:

\[25D + 10Q = 10D + 25Q + 90\]

Simplifying the expression gives:
\[15D - 15Q = 90\]
\[D = 13\]
\[Q = 20 - 13 = 7\]

Key for Investigation 16.2 (page 16-9)

3. A teacher could encourage students to evaluate their proposed values by substituting them into the algebraic equation for a proton (2U + D = +1) and that for a neutron (U + 2D = 0). If both equations balance, then the proposed values are correct. If not, one or both proposed values are incorrect.

4. To solve the equation U + 2D = 0, U cannot equal 1 if D is not equal to - \(\frac{1}{2}\). Only - \(\frac{1}{2}\) + (- \(\frac{1}{2}\)) equals -1. No other fraction added to itself will equal -1.

Key for Probe 6.1 (page 16-11)

1. Case I: The variables represent a particular number. Case II: The variables represent a specific unknown. Cases III and IV: The variables represent a general unknown. In the latter case, the pattern generalizer represents a mathematical principle. Case V: The variables represent varying values.
2. (a) The letter A would represent a particular number in a word problem such as: The area of a field is 1200 square yards. If the field is 40 yards long, how wide is the field? (If A = 1200 square yards and l = 40 yards, then w = ?) (b) The variable A would represent a specific unknown in a word problem such as: If a field is 40 yards long and 30 yards wide, what is the area of the field? (If l = 40 yards and w = 30 yards, A = ?) (c) The letter A would represent a generalized pattern when the relationship among the area, length, and width of a rectangle are summarized as a general formula. (d) The variable A would represent a varying value in a problem such as Farmer Herzog has 140 feet of fence, he wants to enclose the largest area possible, what are the dimensions of the largest rectangular area possible? (If l = 40' and w = 30', then A = 1200 sq. ft.; if l = 38' and w = 32', then A = 1216 sq. ft.; if l = 35 and w = 35, then A = 1225 sq. ft.).

Key for Investigation 16.3 (page 16-17)

Problem 5

Let $E =$ Tuska-loosa
$M =$ A Mudpuppy
$Ø =$ An Octogenarian

First Event. Six weights representing the six Mudpuppies could be placed at Position 5 on the left-hand side of the balance, and five weights representing the five Octogenarians could be placed at Position 6 on the right-hand side of the balance ($5M = 6Ø$).

Second Event. If $E =$ the elephant Tuska-loosa, then $E = 3M + 2Ø$. To balance the three Ms on Position 5 and the two Øs in Position 6 on one side of the balance, the other side of the balance would have to have a total value of 27. This could be done by placing three weights at Position 9 to represent the elephant.

Fifth Event. Possibilities include $1E + 2M$ vs. $5M + 2Ø$ (winner), $1E + 1M$ (winner) vs. $6Ø$, and $E + 2Ø + M$ (winner) vs. $5M + 3Ø$. An informal proof for the last contest is described below.

E + ØØ + M MMMM + ØØØ

From the second encounter, $E = MMM + ØØ$ so have Tuska-loosa on the left leave the rope to sit on the sidelines and 3 Mudpuppies and 2 Octogenarian’s on the right do the same. (Equal quantities taken away)

ØØ + M MM + Ø

Pudding
Pit

Now have one Mudpuppy and one Octogenarian leave each side

Ø M

Pudding
Pit

From the first encounter where $6M = 5Ø$, it follows that an Octogenarian must be stronger than one Mudpuppy.

So Ø "M"

Pudding
Pit

The team of Tuska-loosa, two Octagenarian ands and one Mudpuppy win!

Note this problem uses many of the ideas proposed by the NCTM (1989) Curriculum Standards:

1. The problem can underscore the importance of group work and communicating mathematical ideas.
2. The problem can lead to the discovery of algebraic concepts such as balancing equations, inequality relationships, combining the terms, variable representation, and equivalent values.
3. The problem can be solved informally by using a balance beam or a drawing.
4. The Fifth Event is an open-ended problem.
5. Writing up a solution and presenting it to the class involves both written and oral communication.

Key for Investigation 16.4 (page 16-18)

Problem 1. (b) The Algebra-Tiles model illustrated below should help make clear that Camille and Ranzi’s equations are equivalent. (c) Without considering Vicki’s thinking processes (e.g., asking her to justify her formula), it is not possible to evaluate her effort accurately. She may
have used the same strategy as Ranzi but forgotten to take into consideration the 5’ by 4’ section. On the other hand, she may not have considered the dog house a part of the dog pen—a reasonable assumption. (d) Whether the dog pen is defined as including or excluding the dog house, a try-and-adjust substitution strategy should show that an $x$ of 27 feet gives an area less than 1000 square feet (Ranzi’s solution: $x^2 + 9x + 20 = 992$; Vicki’s solution: $x^2 + 9x = 972$). Thus $x$ must be at least 28 feet.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^2$</th>
<th>$5x$</th>
<th>$x$</th>
<th>length = $x + 5$</th>
<th>width = $x + 4$</th>
<th>area = $(x + 5)(x + 4)$ or $x^2 + 5x + 4x + 20 = x^2 + 9x + 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16</td>
<td>20</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Key for Investigation 16.5** (pages 16-20 to 16-22)

**Part II**

1. The rule for Table K is $(n \times 2) + 3$; for Table L, $(\frac{1}{2})^n$; and for Table M, $2^n + 1$ or $2 \cdot 2^n$.

2. Previa’s conjecture is incorrect because it does not work for all inputs. If the input is 0, then according to her conjecture the output should be $(\frac{1}{2})^0$, which is undefined, not 1. Moreover, if the input is 1, then according to her conjecture the output should be $(\frac{1}{2})^2$ or 1, not $\frac{1}{2}$.

**Key for Probe 16.3** (pages 16-27 and 16-28)

**Part I**

1. a. IV; b. II; c. VIII; d. VI; e. III; f. IV; g. I; h. VI.

2. a. I; b. II; c. V; d. VI; e. IV; f. VII; g. III; h. II.

3. All but graphs V and VIII pass the vertical-pencil test (one and only one $y$ value for each $x$ value) and, thus, represent a function.

4. No, because there are two values (a positive and negative square root) for each $x$. The positive or negative square root of $x$ would be a function.

5. The following graphs consist of a straight line that goes through the origin and, thus, represents a direct proportion: I, II, and IV.

**Part II**

3. In the equation $y = -50x + 600$, $y$ represents the amount of debt left; -50, the constant monthly payment of $50; x$, the number of payments made, and 600; the initial debt.