TEACHING TIPS

AIMS AND SUGGESTIONS

Units 14•1 and 14•2: The Teaching and Learning of Geometry

Many people have not consciously or thoughtfully considered how geometry pervades other aspects of mathematics and our everyday lives. Because they do not fully appreciate the importance of geometry, many elementary teachers spend little or no time on the topic. One aim of chapter 14 is to help readers recognize some of the connections between geometry and other mathematical content areas, other school subjects, and everyday life and, thus, the value of teaching geometry.

As a result of their own mathematics instruction, many pre- and in-service teachers have a narrow view of the teaching and learning of geometry. More specifically, they view the topic almost exclusively in terms of memorizing by rote the name of shapes, formulas, and logical proofs. A second aim of chapter 14 is to help readers see that the study of shapes and space can be interesting and even intriguing. A third aim is to help them see that geometric instruction should focus on developing and applying geometric thinking, not simply memorizing content.

Investigation 14.1: Exercising Geometric Thinking (page 14-2 of the Student Guide) was designed to underscore that informal and formal geometric knowledge and thinking can be an invaluable problem-solving aid. For example, students should recognize that drawing a picture can be helpful in solving Problems 2 and 3 (for a detailed discussion, see page 424 of this guide).

Investigation 14.2: Informal Geometry (page 14-5 of the Student Guide) can serve to illustrate that geometry can be fun to explore, can involve creativity and imagination, has many everyday applications, and can be integrated with the construction of art and other content areas. This investigation can also provide an opportunity for students to explore electronic drawing and design programs and see how technology can be integrated into instruction.

A focus of Unit 14•1 is a discussion of the van Hiele model regarding the development of geometric thinking. (See Figure 14.1 on page 404 of this guide for a description of the three levels of this model relevant to elementary instruction.)

Probe 14.A: Reflecting on Geometry Instruction (page 406 of this guide) can serve as a basis for discussing the limitations of the traditional skills approach, particularly as they relate to the van Hiele model. As Part I of this probe illustrates, this approach focuses on memorizing shape names, not on identifying the attributes of shapes, the relationships among a shape's attributes, or the relationships among different shapes. For example, the depicted worksheet reinforces children's misconception that a rectangle and a square are distinct and unrelated shapes.

Part II of the Probe 14.A can help underscore another common difficulty with traditional geometry instruction: Students frequently construct an incomplete understanding of geometric concepts because they see only a single example or a limited range of examples. (See page 426 of this manual for a more complete discussion of this difficulty and the one discussed in the previous paragraph.)

Investigation 14.3: Using Examples and Nonexamples to Induce the Critical Attributes of Geometric Concepts (pages 14-10 and 14-11 of the Student Guide) illustrates the value of using a variety of examples and nonexamples to help students construct a relatively complete and accurate understanding of geometric concepts.

Investigation 14.3 and Investigation 14.4: Explicitly Examining Relationships Among Figures (page 14-12 of the Student Guide) illustrate how geometric instruction can focus on developing geometric thinking in general and Level 2
thinking in particular. Instructors will probably need to underscore that critical attributes are characteristics shared by all examples of a concept. Because many adult students are not accustomed to Level 2 thinking (e.g., using the definition or critical attributes of a concept and logical reasoning to deduce whether a particular item is an example of a concept), these two investigations are particularly important to complete. Many students find the “if it walks and talks like a duck” analogy for this type of thinking (informal deductive reasoning) both amusing and meaningful. However, it is likely that at least some students will struggle with both of these reader inquiries. For example, they may have difficulty conceptualizing a square as a special case of both rectangles and rhombuses (i.e., as having all the critical attributes of each) and the logical implication of these relationships (e.g., that some rectangles can be rhombuses and vice versa).

Many students think of geometry almost exclusively in terms of Euclidean geometry. In the Student Guide, Investigation 14.5: Informal Topology (pages 14-14 to 14-16), Part I of Investigation 14.6: Expanding Spatial Sense (“Shadow Geometry” on page 14-17), and Investigation 14.7: Exploring Transformation Geometry (pages 14-20 and 14-21) can serve to help introduce them to different geometries. This can be both fun and can encourage students to think in terms of a different set of assumptions (i.e., begin to encourage van Hiele Level 4 thinking). Part II of Investigation 14.8: Exploring Two- and Three-Dimensional Geometry (page 14-24) can serve as a basis for informal instruction to solid geometry, which many adult students may not have experienced.

Finally, Parts II and III of Investigation 14.6 (page 14-18) and Part I of Investigation 14.8 (page 14-22 and 14-23) can help promote a geometry sense. Activities in the latter can help adult students recognize that the sum of two sides of a triangle must be greater than its third side and why (Activity 1) and how children can be helped to discover (a) that the sum of the angles of a triangle equals 180° (Activity 2), (b) the Pythagorean theorem (Activity 3), and (c) the formula for computing the number of diagonals of a polygon (Activity 4). (Activity II of Investigation 14.F, on pages 416 and 417 of this manual, is a more structured version of this last activity.) Geometry sense can be further deepened by asking students to complete any of the additional investigations listed on page 405 of this guide.

SAMPLE LESSON PLANS

Project-Based Approach

Using SUGGESTED ACTIVITIES on pages 418 to 420 of this guide as a menu, have small groups of about four students choose a project. Fairly comprehensive coverage of chapter 14 content can be achieved by having the groups choose different projects and reporting on them to the class. Note that Choices 1 to 5 highlight everyday uses of geometry. For example, Choice 5 requires adult students to compose a Home Connections letter that involves linking some aspect of geometry to everyday life at home. Choices 6 to 9 and 14 involve developing a geometry-based lesson or activity. Choice 6, 7, 8, and 10 involves teaching children; Choices 6, 7, 8, 10, and 11 entail assessing them. Choices 12, 13, 15, and 16 require critically evaluating textbooks or other instructional materials. Choice 14 focuses on cataloguing children’s literature relevant to the topic and developing a lesson around one or more such books. Choices 1, 13, 15, and 16 involve adult students with technology.

Alternatively, an instructor can use Activity File 14.1: An Extended and Integrated Unit on Quilting (pages 14-8 and 14-9 of the Student Guide) as the basis for a whole-class project. This rich unit could engage students in several aspects of geometry (e.g., geometric problem solving, creating patterns, tessellation, and symmetry), using technology (e.g., visiting the numerous quilting-related web sites, such as the Quilt Cuddle Room Homepage at http://www.geocities.com/Heartland/Acres/2136/quilt.html to obtain information or using drawing applications to develop designs), and other mathematical content (e.g., combinations and probability). Information on quilting can be obtained by contacting The National Quilting Association, Inc. (P.O. Box 393, Ellicott City, MD 21041-0393; telephone: 410-461-5733; FAX: 410-461-3693) or the American Quilter’s Society (P.O. Box 3290, Paducah, KY 42002-3290; telephone: 502-898-7903; FAX: 502-898-8890).

Single-Activity Approach

Depending on what an instructor wished to emphasize, either a whole class period or a large portion of one could be devoted to any one of the following reader inquiries in the Student Guide: Investigation 14.1: Exercising Geometric Thinking (page 14-2), Investigation 14.2: Informal Ge-

Multiple-Activities Approach

For relatively broad coverage of the topic, an instructor could choose, for example, the following portions of investigations:

1. To underscore the meaning of geometry and its usefulness, students could read the introduction to Investigation 14.1: Exercising Geometric Thinking (page 14-2 of the Student Guide) and solve Problem 2. By drawing a picture, they should readily see that the boys described in the problem cannot safely view the dog. An instructor may wish to emphasize that drawing a picture essentially entails creating a geometric or visual representation of the problem, often making it easier to solve. Instructors may also wish to underscore how geometric or visual representations or models have been used throughout the Student Guide to facilitate understanding of other mathematical content (e.g., an area model of squares and square roots and fraction or decimal multiplication or division).

2. By taking a moment to do Activity V in Investigation 14.2: Informal Geometry (page 14-5 of the Student Guide), an instructor can make the point that elementary-level, particularly primary-level, instruction should focus on informal geometry (enjoyable, hands-on activities that involve artistry or creativity and that are closely connected to everyday life).

3. Completing Part I of Probe 14.A: Reflecting on Geometry Instruction (page 406 of this guide) can serve as a basis for discussing the van Hiele model and emphasizing that geometry instruction should involve fostering geometric thinking as well as geometric content knowledge. To help students understand the differences among the van Hiele levels relevant to elementary instruction, an instructor might find the points and examples described in Figure 14.1 at the top of the next page helpful.

4. Completing Part I of Investigation 14.3: Using Examples and Nonexamples to Induce the Critical Attributes of Geometric Concepts (page 14-10 of the Student Guide) can illustrate one way of prompting Level 2 (informal deductive reasoning). The vast majority of adult students have only a fuzzy understanding of diagonals because they have seen only a few examples of the concept and were never required to define it explicitly. This can be underscored by asking students Question 3 before they have had a chance to examine the examples and nonexamples illustrated. After doing so, most students should be able to induce the critical attributes of a diagonal (a straight line connecting two nonadjacent vertices of a polygon) and recognize that it can be external. (A common misconception is that a diagonal must be inside the figure. Although this is an attribute of some diagonals, it is not an attribute of all of them and, thus, not a critical attribute.)

5. Completing one or more activities in Investigation 14.4: Explicitly Examining Relationships Among Figures (page 14-12 of the Student Guide) can illustrate how drawing Venn diagrams, creating concept maps, or analyzing errors in such efforts can foster Level 2 thinking. Frequently, each of these activities raises questions about the definitions of geometric figures, the relationships among them, and how these relationships can be represented. Activity III, for instance, frequently provokes the questions, "Why is Mark's drawing incorrect?" (He did not leave any place in his figure where rhomboids—parallelograms in which only the opposite sides are equal—could fit) and "How could you use a Venn diagram to represent the relationships among parallelograms, rectangles, rhombi, and squares?" (Rectangles and rhombi could be represented by two overlapping circles; squares, by the overlapping section; and parallelograms, by an outer circle containing the others).

6. By completing several activities from Investigation 14.5: Informal Topology (pages 14-14 to 14-16 of the Student Guide), an instructor can introduce students to the interesting topic of topology. For example, Activity 1 can lead to the discovery that neither length nor straightness is conserved during transformations in this geometry. These results can be compared to those of shadow geometry and transformation geometry introduced subsequently. Activity 4 can further help students see that topology provides a different perspective. Activities 6, 7, and 8 involving the
Figure 14.1: The First Three Levels of the van Hiele Model

<table>
<thead>
<tr>
<th>Level</th>
<th>Knowledge of attributes</th>
<th>Connections among shapes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (visualization)</td>
<td>None; focus is on global appearance (e.g., (\square) = a rectangle because it looks like one).</td>
<td>None; (\square) = a “square” because it looks like a square but, when turned onto a corner, it is no longer a square because a (\diamond) looks like a “diamond,” not a square.</td>
</tr>
<tr>
<td>1 (analysis)</td>
<td>Focus on identifying attributes of a shape but defines the shape in terms of both critical attributes (those shared by all examples) and noncritical attributes (those shared by only some examples). An example of the latter is defining a rectangle as having four sides, two long and two short. Does not yet recognize how attributes are interrelated.</td>
<td>None; a square is not a rectangle because it does not share the noncritical attributes of two long and two short sides.</td>
</tr>
<tr>
<td>2 (informal deduction)</td>
<td>Identifies a shape by its critical attributes and recognizes relationships among the attributes (e.g., rectangle is a parallelogram with a right angle; a parallelogram with one right angle logically means all four are right angles).</td>
<td>Uses critical attributes of a shape to identify whether or not a particular item is an example or nonexample of the shape and, thus, can recognize relationships among &quot;different&quot; shapes. For example, because a square has all the critical attributes of a rectangle, it is a rectangle. (If it walks and quacks like a duck, then it is a duck.) A square is a special kind of rectangle, because it has an additional critical attribute (all its sides are equal).</td>
</tr>
</tbody>
</table>

A curious Möbius strip can illustrate how entertaining the topic of topology can be. Activity 10, which involves the famous four-color map problem, illustrates a practical application of the field in an engaging way.

7. Completing selected activities of Investigation 14.6: Expanding Spatial Sense (pages 14-17 and 14-18 of the Student Guide) can illustrate how a variety of worthwhile tasks can engage children’s geometric thinking in an entertaining manner. Activity 1 in Part I can serve to illustrate another type of geometry, “shadow geometry,” in which length is not conserved but straightness is. Activities in Part II can serve to introduce students to Tangrams, pentominoes, and tessellation. Part III involves various geometric-reasoning tasks, which even college-level students find challenging. Task 2, for example, illustrates how a little knowledge, namely that a diagonal bisects a rectangle, and deductive reasoning can help solve what appears to be an unsolvable problem.

8. Completing selected activities in Investigation 14.7: Exploring Transformation Geometry (pages 14-20 and 14-21 of the Student Guide) can serve to introduce students to yet another type of geometry, namely transformation geometry. Question 1 of Activity I, for example, can illustrate that in this type of geometry both length and straightness are conserved during transformations.

9. Investigation 14.8: Exploring Two- and Three-Dimensional Geometry (pages 14-22 to 14-24 of the Student Guide) can involve students in several interesting discoveries and illustrate how children can be helped to rediscover, for example, that the sum of the angles in a triangle is 180˚ or that, in a right triangle, \(a^2 + b^2 = c^2\) (the Pythagorean theorem). Note that by answering Question 2 of Activity 2, students can discover that the sum of the interior angles for 4-, 5-, and 6-sided figures is 360˚, 540˚, and 720˚, respectively, and reinvent the formula \(S = (n - 2) \cdot 180\), where \(S\) = sum of the angles and \(n\) = number of sides.

**SAMPLE HOMEWORK ASSIGNMENTS**

Read: Chapter 14 in the Student Guide.
Study Group:

- **Questions to Check Understanding:** 1, 5, 7, 8, 10, and 11 (pages 420 to 422).
- **Writing or Journal Assignments:** 1, 2, and 3 (pages 422 and 423).
- **Problem:** Is It Right? (page 423).
- **Bonus Problem:** A Garden Plot (Problem 3 on page 14-2 of the Student Guide).

**FOR FURTHER EXPLORATION**

**ADDITIONAL READER INQUIRIES**

**Probe 14.A** (page 406)

*Reflecting on Geometry Instruction* involves analyzing a geometry worksheet in terms of the van Hiele model of geometric thinking and self-testing of four geometric concepts. The aim is to underscore two common limitations of traditional instruction.

**Investigation 14.A** (pages 407 and 408)

*The Königsberg Bridge Problem* is a structured discovery-learning investigation of a famous topology problem.

**Investigation 14.B** (pages 409 and 410)

*More Explorations of Transformation Geometry* is an extension of Investigation 14.7 in the *Student Guide*. Studying transformation geometry can provide an avenue for developing the concept of congruence. Exploring the properties of different transformations can lead to an informal definition of congruence as figures with same size and shape. That is, as children experiment with slides, turns, and flips and create new figures the same size and shape as the original figure, the term congruent can be introduced as a succinct way of describing this property of these transformations. Later, congruence can be defined more formally and precisely in terms of transformations in the following manner: Two figures are congruent if translating, rotating, or reflecting one results in the other.

**Investigation 14.C** (pages 411 and 412)

*Part I of Exploring Symmetry* introduces the ideas of translation symmetry, rotation symmetry, and reflection (line) symmetry informally with hands-on activities. **Part II** provides an opportunity to explore these ideas in a more formal and explicit manner.

**Investigation 14.D** (pages 413 and 414)

*Special Categories of Two-Dimensional Shapes* can serve as a basis for exploring the following ways of categorizing two-dimensional shapes: convex versus concave, symmetrical versus nonsymmetrical, points versus lines, and supplementary angles versus corresponding angles versus alternate interior angles.

**Investigation 14.E** (page 415)

For the ancient Greeks, the ideal rectangle was 1.6 times as long as it was wide. Questions 1 and 2 of *The Golden Ratio* could be used to help children discover this special or "golden" ratio. As Question 3 suggests, the golden ratio appears in everyday life, including art (e.g., the Place de la Concorde by Piet Mondrian) and architecture (e.g., the Parthenon at Athens). Students interested in art, architecture, or design might be encouraged to examine specimens for examples involving the golden ratio. As Questions 4 and 5 imply, the golden ratio describes other mathematical relationships also.

**Investigation 14.F** (pages 416 and 417)

*More Explorations of Transformation Geometry* can help students discover that the diagonals of different types of quadrilaterals each have a unique set of properties. Activity II illustrates how students can be guided to rediscover the formula for determining the number of diagonals for a polygon.

**QUESTIONS TO CONSIDER**

1. A worksheet accompanying a textbook for junior-high students included problems similar to that described below. A number of students were able to solve the problems like Example 1 but had difficulty with a problem like Example 2 (both examples are depicted on page 418). Why?

   Instructions: In the following visual-thinking activity, each side of the cube below has a different symbol. (Text continued on page 418.)
The van Hiele model identifies five levels of geometric thinking: recognition of shapes (Level 0), identification of properties (Level 1), informal deduction of relationships (Level 2), formal deduction of proofs (Level 3), and consideration of different geometric systems (Level 4). This part of the probe asks you to analyze a worksheet in terms of its effects on children’s geometric thinking. Share your answers to the questions below with your group or class.

1. This workbook activity encourages what level of thinking according to the van Hiele model? Why?

2. What does this kind of instruction “tell” children about the relationship between squares and rectangles?

3. How might the workbook page above be changed so that it could accurately reflect the relationship between squares and rectangles?

Part II: Examining Knowledge of Plane Geometry

To test your knowledge about a sample of concepts in two-dimensional geometry, answer the following questions. Discuss your answers with your group or class. For which concepts assessed below is your understanding incomplete or not explicit? What do these results say about your formal geometry instruction?

1. In the drawing below, circle all the points that are inside the acute angle xyz.

2. Draw in the altitudes for each of the following triangles.

3. Draw in the diagonals for the following figures:

4. Circle any right triangle below.
Investigation 14.A: The Königsberg Bridge Problem

- Topology (Network Theory) - Any number

**The Problem.** Below is a map of the Prussian city Königsberg. Residents of the city had tried to cross the seven bridges on a single walk without recrossing a bridge. Is this possible, even if you don’t have to end at your starting point?

In the network map of Königsberg to the right, the points A, B, C, and D represent the four regions of the town (separated from each other by rivers), the lines represent the possible walk paths.

**Network Theory.** The eighteenth century mathematician Leonhard Eüler tackled the problem. In doing so, he laid the foundation for network theory, one of the most practical topics in topology, with applications to electrical circuitry and economics (Bergamini, 1963).

To solve the Königsberg Bridge problem and understand Eüler's explanation, consider some simpler cases. In which of the following cases is it possible to take an uninterrupted walk so that all bridges are crossed only once (to make a continuous path such that no path is retraced)?
Investigation 14. A continued

Now consider the more complicated cases below. (a) Does increasing the number of points guarantee a solution? (b) Does the number of lines appear to be the key factor? Is a solution possible if (c) no points have an odd number of lines meeting it, (d) only two points have an odd number of lines, or (e) more than two points have an odd number of lines?

Case X

Case XI

Case XII

Case XIII

Case XIV

Case XV

Case XVI

Bulletin Board Idea

Construct a floor plan such as the one shown to the right. (Particularly for younger elementary students, a teacher might want to start with a relatively simple floor plan.) Stick pins can be placed inside each room and on the sides of the door openings. A long string is attached to a pin either inside or outside the floor plan. Students must try to weave the string in and out of the doors so that they go through each door once and only once. Is this possible with the floor plan shown?
Investigation 14.B: More Explorations of Transformation Geometry

- Spatial sense - K-8 - Individually, small groups or whole class

Transformation geometry examines the motion of rigid figures. This investigation describes a sample of informal activities that explore this geometry—activities that students of all ages find intriguing. To see what is involved or to extend your own knowledge of geometry, try the following activities yourself.

Activity I: Grids

**Challenge 1** (3-8). In Grid A below, redraw the figure shown so that the figure is slid over 2 dots to the right.

**Challenge 2** (3-8). In Grid B below, redraw the figure shown so that Point X is in the same place but the rest of the figure is turned clockwise at a square (90°) angle.

**Challenge 3** (3-8). In Grid C below, redraw the figure shown so that the figure is flipped over the line.

**Follow-up.** To check whether the slide, turn, or flip was done correctly, trace the original figure, cut out the tracing, and then move the tracing so that it fits over your drawing.

Activity II: Reflections.

Note that all of the following challenges provide purposeful opportunities to practice enumeration. Challenges 5 and 6 can be used to explore the addition doubles and their odd-even pattern.

**Challenge 1** (K-2): Using the picture below and a mirror, what is the most circles you can make? Be sure to count both circles drawn on the paper and circles reflected in the mirror.

**Challenge 2** (K-2): Using the picture above and a mirror, can you make 2, 4, or 6 circles?

**Challenge 3** (1-3): Using the picture above and a mirror, can you make 1, 3, 5, or 7 circles?

**Challenge 4** (K-2): What is the most circles you can make with a mirror and 2, 6, 8, or 10 circles?

**Challenge 5** (1-3): After children discover that a mirror can double a collection, ask them whether doubling an even number, such as 2, 4, 6, 8, or 10, always, sometimes, or never results in an even number?

**Challenge 6** (1-3): If a mirror is used to double an odd number of circles, such as 3, 5, or 7 circles, will the result always, sometimes, or never be an even number?

Activity III: Waxpaper Activities. Consider the following problems:

- **A Waxpaper Solution?** (1-8). Mr. Beemish had short wooden rods that he needed to cut in half for a craft project. Unfortunately, he could not find his ruler and did not want to drive to a store. His wife recalled using waxpaper to divide a line into equal lengths but could not remember exactly how to do it. How could Mr. Beemish use waxpaper to determine the midpoint of his rod?

**Extension.** How could Mr. Beemish use a compass (or a pencil tied to string) to bisect the rods?
Investigation 14.B continued

- **A New Waxpaper Application** (3-8).
  Two rods of a craft project met at an angle. Mr. Beemish wanted to place a decal exactly between the two rods. How could he use waxpaper to bisect the angle made by the rods?

- **More Waxpaper Applications** (3-8).
  Mr. Beemish wanted to center pictures inside differently shaped wooden frames. How could he use waxpaper to determine the center of (a) a rectangular-shaped frame, (b) a circular-shaped frame, and (c) a triangular-shaped frame?

**Activity IV: Escher-Type Tessellations.** M. C. Escher used transformations and tessellations to create well known works of art involving repeated and interlocked shapes. Students can create their own Escher-Type tessellations in two ways.

1. **Using a translation.**
   - Begin with a simple shape
   - Make a change to the opposite side of the shape
   - Tessellate the new shape

2. **Using a rotation.**
   - Begin with a simple shape
   - Make a change to one half of a side
   - Rotate the change at the midpoint to the other half of the side
   - Tessellate the new shapes

**Extensions**

Note that changes involving translations or rotations can involve curved lines as well as straight lines as shown above. Consider the example below.

Moreover, a simple shape can be changed by more than one transformation or by a combination of translations and rotations. Experiment with changing combinations of transformations. Then try to create a bird tessellation or a tessellation of your own design.

Collect examples of Escher-type tessellations.

*Books such as “The World of M. C. Escher” (grade 7 and up) and “3-D models (M. C. Escher Kaleidocycles),” videos such as “The Fantastic World of M. C. Escher” (grade 7 and up), CDs such as “Escher Interactive: Exploring the Art of the Infinite,” and prints (e.g., “M. C. Escher: 29 Master Prints”) are available from publishers such as Cuisenaire-Dale Seymour Publications.*
Investigation 14.C: Exploring Symmetry

Informal and more formal explorations of symmetry • K-8 • Any number

Symmetry is a key characteristic of many everyday things. Part I includes activities that explore this concept informally; Part II, activities that explore it more formally and explicitly. To see how and to perhaps deepen your understanding of symmetry, try the activities yourself.

Part I: Informal Explorations

Activity 1: Potato Prints (◆ 2-6). Cut a potato in half. Cut a design into the potato surface (see Figure A below). Dip this cut surface in tempera and make a row of prints of your design (see Figure B).

![Figure A: Potato engraving](image1)

![Figure B: Row of printed designs](image2)

Activity 2: Potato Print Challenge (◆ 2-6). Try to cut a design in a potato that will come out the same even if you turn the potato half way around (rotate it 180°).

Activity 3: Folding Paper Shapes (◆ 2-6). Cut out shapes like those shown at the top of the next column. Try folding each shape so that two halves match. For each shape, how many different ways can you do this? Were there any shapes that you could not fold in half to create a match? Complete the chart in the next column. Discuss your findings with your group or class. What conclusions can you draw?

<table>
<thead>
<tr>
<th>Shape</th>
<th>Number of ways figure can be folded into two matching halves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td></td>
</tr>
<tr>
<td>Isosceles triangle</td>
<td></td>
</tr>
<tr>
<td>Right triangle</td>
<td></td>
</tr>
<tr>
<td>Parallelogram</td>
<td></td>
</tr>
<tr>
<td>Rectangle</td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td></td>
</tr>
<tr>
<td>Regular pentagon</td>
<td></td>
</tr>
<tr>
<td>Circle</td>
<td></td>
</tr>
</tbody>
</table>

Activity 4: Cut Outs of Folded Shapes (◆ 1-8). Fold a piece of paper in half. Extending from the fold, draw a design (e.g., see Figure C below). Cut the shape out and unfold the paper.

![Figure C](image3)

Activity 5: Paper Trains (◆ 3-8). An extension of Activity 4 leads to the following challenging problem:
Study 14.C continued

- **Paper Train.** How can paper be folded so that drawing and cutting out a design produces a train of designs such as the one shown in Figure D below?

![Paper Train Design](image)

**Part II: More Formal Explorations**

Corresponding to the three basic motions are three types of symmetry:

1. **Slide (translation) symmetry.** A row of potato prints (Activity 1 in Part I) is an example of a tessellation with translation symmetry. The design in Figure B (on the previous page) could be seen as repeatedly making a translation of a form. More formally, a design has translation symmetry if there is a translation of a figure that creates identical figures.

2. **Turn (rotation) symmetry.** Creating a design that will look the same even if turned upside down (e.g., Activity 2 in Part I) is an example of rotational symmetry. An equilateral triangle is another example, because turning it 120° yields an identical-looking triangle. More formally, a design has rotational symmetry if it can be rotated less than one complete turn to create an identical design.

3. **Flip (reflection or line) symmetry.** Designs that can be folded in half to create matching figures (Activities 3 and 4 in Part I) are examples of reflection symmetry. More formally, a design has reflection or line symmetry if there is a reflection that creates identical figures.

The Venn diagram in the next column suggests that some figures do not have any kind of symmetry (region A), that some have only one kind of symmetry (e.g., region B includes designs that have only line symmetry), that some have multiple symmetries (e.g., region C includes designs with both reflection and translation symmetry). Examine the block letters and block letter designs below. For each, indicate the letter of the Venn diagram region where the example would fit. For examples with reflection symmetry, indicate how many lines of symmetry it has.

1. M 3 O 5. SSSS 7. QQQQ

**Questions for Reflection**

1. Where would the following design fit into the Venn diagram above and why?

![Venn Diagram](image)

2. Analyze block letters of the alphabet. Which have transformation symmetry? Which have rotation symmetry? And which have reflection symmetry? Which letters have more than one line of symmetry?

3. (a) The paper train illustrated in Figure D (in the upper left-hand corner of this page) has what kind of symmetry or symmetries? (b) Would it be possible for paper trains to illustrate all three types of symmetries? If so, illustrate with an example. If not, explain why.
Investigation 14.D: Special Categories of Two-Dimensional Shapes

- Exploring different ways of categorizing two-dimensional shapes

This investigation includes a sample of activities for informally exploring special categories of two-dimensional shapes. To see what is involved and to perhaps extend your own understanding of these categories, try the following activities yourself.

Activity I: Convex versus Concave (☆ 6-8 ☆ Any number)

Choose any two points inside the convex shapes below and connect them with a straight line. Do this a number of times for each shape. Do the same for the concave shapes. In what way are all convex shapes alike? In what way do they differ from concave shapes?

Convex

![Convex Shapes]

Concave

![Concave Shapes]

Teaching Tips. Intuitively, convex can be thought of as an undented shape and concave, as a dented shape. Level 0 children may use these unstated criteria to sort shapes by their appearance. Encourage them to give their own name to these categories (e.g., smooth and dented). In time, students can learn the formal terms convex and concave. The latter can be likened to a cave, which goes in.

Activity II: Symmetry (☆ 6-8 ☆ Any number)

This more formal investigation could build on the informal exploration of symmetry described in Part I of Investigation 14.C (on pages 411 and 412 of this guide). Which of the figures listed below have reflection symmetry? Which have rotation symmetry? Which have both? Which have neither? Complete the table below.

<table>
<thead>
<tr>
<th>Reflection Symmetry</th>
<th>Rotation Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes or No/Number of lines of symmetry</td>
<td>Yes or No/Number of possible rotations*</td>
</tr>
<tr>
<td>scalene triangle</td>
<td></td>
</tr>
<tr>
<td>isosceles triangle</td>
<td></td>
</tr>
<tr>
<td>equilateral triangle</td>
<td></td>
</tr>
<tr>
<td>parallelogram</td>
<td></td>
</tr>
<tr>
<td>rectangle</td>
<td></td>
</tr>
<tr>
<td>rhombus</td>
<td></td>
</tr>
<tr>
<td>square</td>
<td></td>
</tr>
<tr>
<td>regular pentagon</td>
<td></td>
</tr>
<tr>
<td>regular octagon</td>
<td></td>
</tr>
<tr>
<td>circle</td>
<td></td>
</tr>
</tbody>
</table>

*Excluding complete (360°) turns.

Teaching Tip. A natural place to have children look for symmetries is faces, leaves, and other elements of nature.
Investigation 14.D continued

Activity III: Properties of Points and Lines (✦ 7-8 ✦ Any number)

1. How many straight lines contain both points $A$ and $B$ in the plane below? Is this true for any two points in a plane?

2. A line can be thought of as a copy of the real number line. What is the distance from point $A$ to point $B$ in terms of the coordinates $(0, a, b, c \ldots)$ on the number line shown below it? From point $A$ to point $E$? From point $B$ to point $D$? What conclusion can you draw about determining the distance between any two points on a line?

3. Point $P$ is not on line $l$ to the right. How many straight lines in the plane can be drawn through $P$ that are parallel to line $l$? What general conclusion can you draw about such situations?

Activity IV: Supplementary, Corresponding, and Alternate Interior Angles (✦ 5-8 ✦ Any number)

1. After students understand that any angle whose rays form a straight line measures 180°, pose a problem such as the following:

   - **Angled Streets.** Melrose Place is a straight street that is intersected by another straight street, Fashion Road. (a) If angle $A$ below is 80°, what are angles $B$, $C$, and $D$? (b) Angles $A$ and $B$ are called supplementary angles. What other supplementary angles are formed by the intersection of Fashion Road and Melrose Place? (c) Propose a definition of supplementary angles and discuss it with your group or class.

2. Soap Avenue runs parallel to Fashion Road. (a) Use a protractor to measure angle $E$. (b) Now determine what angles $F$, $G$, and $H$ are without the protractor. (c) What relationship, if any, do angles $E$, $F$, $G$, and $H$ have with the corresponding angles $A$, $B$, $C$, and $D$, respectively? (Angle $A$ and angle $E$ are corresponding angles because they are in the same relative position.) Hope Avenue is not parallel to Fashion Road or Soap Avenue. Measure angle $I$. (e) Now determine what angle $J$, $K$ and $L$ are without a protractor. (f) What relationship, if any, do angles $I$, $J$, $K$, and $L$ have with their corresponding angles $A$, $B$, $C$, and $D$? (f) Just east of Hope Avenue is Lonely Street. What must be true about angles $M$, $N$, $P$, and $O$ if Lonely Street is parallel to Fashion Road and Soap Avenue? (g) What must be true about angles $M$, $N$, $P$, and $O$ if Lonely Street is not parallel to Hope Avenue?

3. In the figure above, angles $D$ and $F$ are called alternate interior angles because each is at a different intersection on Melrose Place, both angles are between the lines Fashion Road and Soap Avenue, and they are on opposite sides of the line Melrose Place. (a) Identify other alternate interior angles in the figure. (b) What must be true about alternate interior angles if two lines are parallel?
Investigation 14.E: The Golden Ratio

- Ratios and measurement in everyday life • 6-8
- Individually or in small groups + class discussion

This activity explores a special ratio called the golden ratio. It had special significance to the ancient Greeks and is still of importance today in such areas as art and architecture. To learn more about the golden ratio, try the activity yourself, preferably with the help of your group. Discuss your findings with your group or class.

1. In each row, circle the "most-pleasing" rectangle—that which best fits your image of an ideal rectangle. Compare your choices to those of others in your group or class. What do most people view as the most pleasing?

2. The ancient Greeks had a definite idea about what constituted a most-pleasing rectangle. In their view, the ratio of the longer side to the shorter side was the same for all most-pleasing rectangles. What is the ratio of the longer side's length to the shorter side's length for each of your group's most-pleasing rectangles (rounded to tenths)? Are all the ratios the same? If so, what is it (rounded to tenths)? If not, consider which of the rectangles above do have the same ratio of longer side to shorter side and determine what this ratio is.

b. An architect is designing a store front which will take the form of a rectangle. If the width of the store must be 40 feet, what should the height be to have the most pleasing appearance?

3. a. An artist wants to paint a rectangle that is 6 cm. wide and that is pleasing to the eye. How long should the artist make the rectangle?

b. An architect is designing a store front which will take the form of a rectangle. If the width of the store must be 40 feet, what should the height be to have the most pleasing appearance?

4. The ratio of the longer side of the most-pleasing rectangle below to the shorter side \( \left(\frac{5}{3}\right) \) is equal \( \frac{5}{3.1} \) or 1.6. If a longer side is combined with a short side to form a straight line, then \( L+S \) (half the perimeter) is \( 5 + 3.1 \) or 8.1. (a) What ratio involving \( L+S \) and some element of the rectangle also equals 1.6? (b) Write a proportion that involves \( L, S, L+S, \) and some element of the rectangle.

5. Use a centimeter ruler to measure the diagonal \( \overline{AD} \) and a side of the regular pentagon shown below. What is the ratio of the diagonal and the side?
Investigation 14.F: Delving Into Diagonals

◆ Discovering geometric relationships ◆ 3-8 ◆ Small groups of four

This investigation involves exploring diagonals and finding patterns and relationships. To see what is involved and to perhaps deepen your own understanding of the concept, try the following activities yourself. Discuss your findings with your group or class.

Activity I: Using Dot-Matrix Paper or Geoboards to Explore the Diagonals of Quadrilaterals.

1. For each figure below, draw in the diagonals. Note that instead of dot-matrix paper, students could use geoboards. In what way do the diagonals of different types of quadrilaterals differ?

A. Irregular quadrilateral  B. Parallelogram  C. Rectangle

D. Kite  E. Rhombus  F. Square

2. a. For which quadrilaterals are the diagonals perpendicular?
   
b. For which are the diagonals equal in length?
   
c. For which do the diagonals bisect each other?

3. On dot paper, try additional examples of each type of quadrilateral. Try different sizes and where possible different shapes. (For example, whereas irregular quadrilaterals come in various shapes, all squares have the same shape—are similar) Do your observations in Question 2 hold up?

4. Summarize the data in a table. Does each type of quadrilateral have a unique set of relationships between its diagonals?

Activity II: A Many-Diagonals Problem

1. Try solving the following problem using the drawing below. What do you discover?

■ The Number of Diagonals in a 12-Sided Polygon (Dodecagon). How many diagonals can a dodecagon have? A dodecagon can be constructed from wooden pattern blocks in the following manner:

2. Drawing in the diagonals of a dodecagon can quickly become a tedious chore. Moreover, the completed diagram would be so busy, it would be difficult to count all the diagonals accurately. This is a problem where finding a pattern would be most helpful.
3. There are 12 vertices and nine diagonals from each vertex. It is tempting to conclude that there are $12 \times 9$ or 108 diagonals. However, if diagonals are counted just once, do all the vertices have the same number? To answer this question and solve the original problem, consider some simpler cases. For each polygon below, list all the diagonals that can be drawn from each vertex, cross off any repetitions, and then total the number of new diagonals from each vertex. Are there any patterns that would help you determine the number of diagonals in other polygons such as a dodecagon?

<table>
<thead>
<tr>
<th>Polygon</th>
<th>List of diagonals from each vertex</th>
<th>Total</th>
<th>Total new (without duplicates)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 sides</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 sides</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>A:</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>B:</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>C:</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>D:</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>E:</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Total:</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>6 sides</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>F:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 sides</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>F:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>G:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. To devise a shortcut for determining the number of diagonals for a figure, compare the number of sides of a figure to the number of diagonals that can be drawn from each vertex. What pattern do you detect? Now consider how you could determine the total number of diagonals. What do you then have to do to this total to determine the number of “new” (nonduplicate) diagonals? Summarize your shortcut as an algebraic equation.

5. Another way to approach the problem is to construct a table and look for a pattern.

<table>
<thead>
<tr>
<th>number of sides in a polygon</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of diagonals from each vertex</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total number of diagonals</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total new diagonals</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the space provided, show what symbol should be on the bottom of the cube after the second turn.

1. Draw in the three altitudes of the following triangles. With triangle c, extend the altitudes in both directions. What do you notice about the altitudes in all three cases?

2. (a) Draw a concept map that includes the following concepts: cones, cubes, cylinders, polyhedra, prisms, pyramids, and solids. Make sure to explicitly label each connection. (b) Draw an example of each concept and include these examples in your concept map.

SUGGESTED ACTIVITIES

1. (a) Collect newspaper, magazine, or web-site articles that illustrate the everyday use of geometry. (b) Identify the type of geometry (e.g., topology, shadow geometry, transformation geometry, Euclidean geometry) involved in each case. (c) Identify key geometric terms used in each and evaluate the accuracy of the definitions given or implied. (d) Detail how you could use the material collected for a bulletin-board or learning-center display appropriate for a grade level of your choice.

2. (a) Describe your own idea for a bulletin-board display involving geometry. Indicate the learning aims of the display and illustrate how it might look. (b) Implement your bulletin-board plan.

3. Make a catalog of geometric models or analogies useful for teaching mathematics concepts or skills at the primary or intermediate level.

For each example, illustrate the model and briefly explain how or why it might be helpful.

4. Make or collect needlework mats, napkins, tablecloths, or other handmade crafts that involve various types of symmetries. If this is not practical, begin a collection of pictured items that do. Identify those items that involve tessellation.

5. Devise a Home Connections letter that describes two or more geometry activities that parents or guardians can do with children at home. Gear the activity to the developmental level of the children you plan to teach. An example is illustrated below.

Home Connections

Dear Parent or Guardian:

Our class has been studying different types of symmetries. Would you ask your child to explain slide symmetry, turn symmetry, and flip (or mirror) symmetry to you? Would you then help your child to find two examples of each in your house. Please have your child draw a picture, describe, or otherwise record each example.

Thank you for your help.

6. (a) Activity File 14.1 (pages 14-8 and 14-9 of the Student Guide) illustrates the investigative approach. Using this as a guide, develop an inquiry-based lesson plan that would naturally lead to the teaching of geometric concepts or skills. Specify the intended grade level and the problem (or other worthwhile task) that will prompt the investigation. Describe how children might try to solve the problem informally. Indicate what mathematical processes (e.g., problem solving, deductive reasoning, communicating, conjecturing) will be involved in the lesson. Indicate what geometry concepts or skills children will need to learn or practice in order to solve the problem. Indicate also how you would help children learn new concepts and skills in a meaningful fashion. Consider what foundational concepts and skills children would need in order to learn the new geometry content and to be able to devise a solution strategy for the problem. (b) Implement your lesson with an elementary-level class. Assess your pupils and evaluate your lesson. (c) Share your plan and what you have learned with your class.
7. (a) Devise a nature lesson that is based on symmetry. One possible topic is the role of symmetry in animal attractiveness. For example, a female scorpion fly can sense whether a male's wings are matched and will not settle for a suitor with asymmetrical wings. Likewise, female crickets seem to prefer the serenade of suitors with perfectly matched legs. (b) Try out and evaluate your lesson.

8. (a) Devise a lesson that involves one of the reader inquiries in chapter 14 of the Student Guide. (b) Try it out and, if possible, videotape your lesson with a developmentally ready group or class of intermediate-level children. Assess your pupils' learning. Evaluate the strengths and weaknesses of your plan and lesson. (c) Share with your class a summary of your lesson plan and the results of your teaching experience.

9. Create your own Venn-diagram activity on symmetry like that in Part II of Investigation 14.C (on page 412 of this guide). Use a different set of examples—either different letters or nonletter designs. (Note that this would be an excellent open-ended question to assess students' understanding of transformation, rotation, and reflection symmetry.)

10. (a) Tutor a group of elementary-level pupils for at least one one-hour session per week over the course of at least two months on geometry. (b) Document your tutees' progress in content knowledge and geometric thinking. In particular, describe any informal strategies they invented to solve problems and their symbolic-level shortcuts. (c) Present your results and conclusions about them to your class.

11. (a) Assess the level of geometric thinking of several children at grades 4, 6, and 8. Use Activity File 14.A* to the right as a basis of your evaluation. (b) Find or devise additional tasks that will help you assess children at the grade level you hope to teach.

12. (a) Examine a recent textbook at a grade level of your choice. Analyze the geometry instruc-

*Based on a task described in “The van Hiele Model of Thinking in Geometry Among Adolescents” by D. Fuys, D. Geddes, and R. Tischler. Journal for Research in Mathematics Education, Monograph No. 3, © 1988 by the National Council of Teachers Mathematics, Reston, VA.


- Using relationships to define geometric concepts in terms of critical attributes
- Whole class, small groups, or individually

Either in the context of a class discussion or as the basis of a game, a teacher can pose a question such as, What is the fewest properties needed to define a rectangle? To focus the class discussion or make the game easier, the teacher can list properties from which the students can choose. This activity can prompt a discussion of how properties are related and, thus, which are critical to defining a geometric concept and which are not. Played as a game, children can work together in teams of four. After recording their list of properties, the teams can compare, justify, and discuss their lists. The shortest list may not be the best if it leaves out a critical attribute and, thus, results in an overly broad definition. Long lists may be overly restrictive or contain noncritical attributes. Note that this task can also be used to gauge students' level of thinking either individually or as a group.

An Example of Level 0 Thinking. Becker examined an example of a rectangle and concluded, "The top and bottom are the same size and the two sides are the same.” Note that, although he described obvious properties of a rectangle, his definition is not sufficiently complete to distinguish it from other quadrilaterals.

An Example of Level 1 Thinking. Arsenio examined an example of a rectangle and began to list the properties. "Four sides, but that’s true of any quadrilateral. Opposite sides equal and parallel. Also true of parallelograms though. Four square angles.” Note that Arsenio adds characteristics until he has enough to distinguish a rectangle from other shapes. However, he does not see relationships among the characteristics.

An Example of Level 2 Thinking. Kagen reasoned, “It has four sides so it must have four angles. The angles are right angles so you don’t have to say all the angles are equal. The sides are parallel, so that opposite sides must be equal. A rectangle is a four-sided figure with opposite sides parallel and a right angle.” Note that this child used deductive reasoning to eliminate unnecessary properties.
tion. Is it consistent with the general guidelines outlined in Subunit 14•2•1 of the Student Guide? Specifically, is it integrated with instruction from other mathematical topics? Does it focus on the development of geometric thinking? What van Hiele levels of geometric thinking does it encourage and how? Does the textbook include informal geometry activities? Identify whether the suggested instruction best fits the description of a skills, conceptual, investigative, or problem-solving approach.

(b) Present your findings and conclusions to your class using appropriate graphs and statistics.

13. (a) Using publishers’ catalogues, reference books on teaching, the internet, or other sources, make a list of instructional resources that would be helpful in teaching geometry.

(b) Obtain one or more of the resources. Evaluate a resource in terms of what approach to mathematics instruction it seems to suggest. Indicate how one or more recommended activities could serve, or be adapted to serve, as a worthwhile task and the basis for the investigative approach. (c) Share your data, evaluation, and teaching ideas with your class.

14. (a) Find examples of children’s literature in which geometry plays a key or interesting role. (b) Consider how a lesson or unit that embodies the investigative approach could be built around one or more of the books on your list.

15. (a) Try out one of the examples of educational software listed on pages 14-26 to 14-28 of the Student Guide or an educational program of your own choosing. (b) Evaluate the software in terms of its ease of use and instructional value. (c) Share your evaluation with your class, including a demonstration of particularly interesting or useful aspects of the program review.

16. Visit the Quilt Cuddle Room Homepage at http://www.geocities.com/Heartland/Acres/2136/quilt.html, one of the web sites listed on page 14-26 of the Student Guide, or a web site of your own choice. (a) Evaluate the web site as a resource for elementary-level students. Consider how the web site could provide the basis for a worthwhile task, activity-based lesson, or an integrated unit. (c) Share your analyses of the web site with your class.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. Circle the letter of any of the following statements that, according to the Student Guide, are true.

a. At the Visual Level (Level 0), children learn to recognize geometric shapes by their characteristics (properties or attributes).

b. At the Analysis Level (Level 1), children see how characteristics of a figure are related.

c. At the Analysis Level (Level 1), children do not see how different shapes are related.

d. At Informal Deduction Level (Level 2), children figure out the critical attributes of shapes.

e. Progress through the levels tends to be gradual and sequential.

f. The level of a child’s geometric thinking depends on age and typically does vary across concepts.

g. At Level 1, a child views a rectangle as a special kind of parallelogram.

h. At Level 3, children begin to define figures in terms of a minimal set of critical attributes.

2. (a) Which of the following shapes is most specifically defined (i.e., has the largest number of defining properties or attributes)? (b) Which represents the broadest category (i.e., has the fewest numbers of attributes)?

![Shapes A to E]

3. Figure A below is a trapezoid. Is Figure B? C? D? E? Why or why not?

![Figures A to E]
4. (a) Draw a Venn diagram that illustrates the relationship among a parallelogram, a rectangle, a rhombus (an equilateral parallelogram), and a square. (b) Assess Veronica’s effort in using a Venn diagram to illustrate the relationships among parallelograms (P), rectangles (Rec), rhombuses (Rh), and squares (S).

5. (a) Draw a concept map to illustrate the relationships among the following concepts: circles, enclosed figures, hexagons, parallelograms, polygons, quadrilaterals, trapezoids, and triangles. Include specific linking phrases. (b) Using the letters below to indicate where each of the following 10 examples would fit in the concept map:

6. (a) How many diagonals does Figure A below have and why? (b) How many diagonals does Figure B below have and why? (c) How many diagonals does Figure C below have? (d) How many diagonals does a 16-sided polygon have?

7. Describe all possible lengths of the third side of a triangle with sides measuring 8 cm and 14 cm.

8. To practice inductive and deductive reasoning, Miss Brill had her class do the following activity.

These are Blurs

These are not Blurs

Which of the following is a Blurb?

(a) Answer the question above. (b) Has Miss Brill clearly illustrated the concept of Blurb? Why or why not?

9. Webster’s New Twentieth Century Dictionary (unabridged 2nd edition) defines rhomboid as a
A parallelogram with oblique angles and only opposite sides equal. (a) What is an oblique angle? (b) Circle any figure below that, according to this definition, is a rhomboid. (c) Would it have made any difference if the dictionary had defined a rhomboid as "a parallelogram with an oblique angle and only opposite sides equal"?

10. Circle the letter of any of the following statements that, according to the Student Guide, are true.

a. Informal geometry is appropriate for primary-level students but not intermediate-level students.

b. Geometry instruction should focus on learning the names of geometric shapes.

c. Primary-level children should be encouraged to draw and to sketch three-dimensional shapes.

d. In topological transformations, straightness and length are preserved.

e. In projective transformations, straightness, but not length, is preserved.

f. In transformation geometry, neither straightness or length is preserved.

g. Topology should first be introduced in the middle-school grades.

h. Slides provide a concrete analogy for translations; flips, a concrete analogy for reflections.

i. Transformation geometry should begin with an investigation of body motions.

j. Two figures are congruent if and only if sliding (translating) one results in the other.

k. A discussion of points and lines should be the basis for initial lessons on two- and three-dimensional geometry.

l. Considering geometries beyond three-dimensions or the relationships between two- and three-dimensional geometry is too abstract for elementary-level children.

11. Circle the letter of any of the following statements that, according to the Student Guide, are true. Change the underlined portion of any false statement to make the statement true.

a. Geometry is a particularly difficult topic to relate to everyday life.

b. Critical attributes of a parallelogram include four sides, opposite sides parallel, adjacent sides unequal, and opposite sides equal.

c. Critical attributes of a rhombus are a parallelogram with equal sides and no right angles.

d. A triangle has three diagonals; a square; four, and a circle, an infinite number.

e. Transformation geometry can be thought of as the "geometry of distortion," "balloon geometry," or "rubber-sheet geometry."

f. Tessellation involves covering a surface with a single shape in a repeating pattern with no gaps.

g. Euclidean geometry can be thought of as the geometry of motionless and constant shapes.

h. Miras or Reflectas are useful in exploring slides.

WRITING OR JOURNAL ASSIGNMENTS

1. Mr. Muiri's textbook defines a square as a figure with four sides, all sides the same size, and four square corners. It defined a rectangle as a figure with four sides, two long sides, two short sides, and four square corners. (a) What van Hiele level of geometric thinking do these definitions encourage? Briefly justify your answer. (b) Evaluate these definitions in terms of the criti-
cal attributes of a square and a rectangle. (c) Justify your answer in terms of what it implies about the relationship between a rectangle and a square.

2. Asked to define a rectangle in terms of the fewest properties possible, Dora suggested, "A four-sided polygon with four square angles." (a) Is Dora’s definition satisfactory? That is, does it distinguish a rectangle from nonrectangles? Why or why not? (b) From her answer, is it clear what van Hiele level of geometric thinking she used? Briefly explain why or why not?

3. Mr. Loftus’ class had just agreed on a definition of a square and were trying to find examples and nonexamples of this concept. Arla’s group proposed a diamond as a nonexample. Demi’s group disagreed. According to the Student Guide, how could Mr. Loftus help his class resolve this conflict?

4. Miss Brill found that her mathematics-for-elementary-teachers textbook defined a trapezoid as a quadrilateral with exactly one pair of parallel sides. The text specified that a parallelogram is not a trapezoid. However, her math methods textbook defined a trapezoid as a quadrilateral with at least one pair of parallel sides. Frustrated that something as simple as the definition of shape could be so confusing, she called the reference desk of the local library. The librarian checked an authoritative source: The Mathematics Dictionary (James & James, 1949). Trapezoid was defined as: "a quadrilateral which has two parallel sides. It is usually required that the other sides be nonparallel" [italics added].

The next morning, Miss Brill visited the high school’s mathematics department. Miss Thompson listened with interest and commented: "Your question underscores the importance in mathematics of communicating precisely—of defining and using terms carefully." The high school teacher then drew the figure below and described a theorem for isosceles trapezoids, "In an isosceles trapezoid, the adjacent angles a and b are equal." Miss Thompson continued: "If this theorem is true and a parallelogram is a trapezoid, what can you conclude about all parallelograms? What are the implications of your conclusion?

5. (a) Miss Brill read an article about geometry in a journal for teachers in which the authors recommended defining a rectangle as an oblong figure with opposite sides parallel and four right angles. Evaluate this teaching recommendation. (b) In an article titled "Getting in Touch with Geometry," Kelly and Kelly (1988) defined a rhombus as a quadrilateral with equal sides, two obtuse angles and two acute angles. Is this definition of a rhombus correct or incorrect? Justify your answer in terms of what a teacher should focus on when teaching geometric concepts and what the definition implies about the relationship between a square and a rhombus.

PROBLEMS

■ Is It Right? (7-8)

Is the triangle in the geoboard depicted below a right triangle? Justify your answer. Hint: If $a = 1$ unit, what must be the value of $b$?

■ A Garden Plot Revisited (7-8)

Recall Problem 3 of Investigation 14.1 on page 14-2 of the Student Guide. If Francesca planted her strawberry plants in a triangular lattice as shown below, how many strawberry plants would she be able to plant in her 8-feet by 10-feet garden?
ANSWER KEY for Student Guide

Key for Investigation 14.1 (page 14-2)

Problem 1

One informal solution method for determining the length of the third side of a right triangle with sides of 3 and 4 linear units is to mark off the length of the side labeled 4 and then, using the same starting point, mark off the length of the side labeled 3. The result, as illustrated below, serves to define one linear unit:

\[ B = 4 \]
\[ C = 3 \]
\[ B - C = 1 \]

Problem 2

No, they cannot safely view the dog. As the figure below makes clear, Rover can reach the last 2 feet of the walkway, which makes reaching the back corner of the garage perilous.

Questions for Reflection

1. Drawing a picture might be helpful.

2. (a) In the investigative approach, a teacher would encourage students to resolve the discrepancy themselves. (b) Alexi was incorrect. As Alison explained to him, “We measured [the plot] out first, then planted the plants. You planted the plants first—you numbered the plants. Here’s one foot (drawing a line). One plant can go here (points to the left end of the line) and one can go here (points to the right end of the line). You only have one plant in one foot. Why don’t you put two?”

3. (a) This problem provides an opportunity to connect a realistic measurement situation to multiplication. It illustrates a situation where multiplication provides a shortcut for the time-consuming process of counting a rectangular array of items. (b) Alison’s reluctance to count the first item in the first row is understandable and probably common. It is based on an incomplete understanding of multiplication. With a groups-of interpretation, one factor represents the number of groups and the other factor represents the number of items per group. This can, perhaps, be more easily shown with a smaller expression such as \( 4 \times 3 \):

\[ \text{count to determine the number of items per group} \]
\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array} \]

Note that if the first item of the first row is not counted in both counts, either number of items per group or the number of groups will be one less than it should be.

4. By experimenting with sketches or applying geometric reasoning, it is possible to come up with a better solution to A Garden Plot (Problem 3) than Arianne’s. To make maximum use of the garden space available, Francesca should plant her strawberry plants in a triangular lattice (as shown below) instead of using a square lattice. (This helps to explain why some orchards and crops are planted in diagonal rows.) See the key for A Garden Plot Revisited on page 430 of this guide for more information.
Key for Investigation 14.3 (pages 14-10 and 14-11)

Part I: Diagonals

1. A diagonal is a line that extends between the vertices of any two nonadjacent angles in a polygon (or polyhedral).

2. None. There are no nonadjacent angles in a triangle.

3. $\overline{BD}$, $\overline{CG}$, $\overline{DF}$, and $\overline{EG}$.

Part II: Trianquads

1. From the examples and nonexamples given, it is not possible to tell whether or not Figures a, b, and c are trianquads or not. All the examples of trianquads included a triangle and quadrilateral connected at a single vertex. Thus, it is not clear whether a triangle and a quadrilateral connected at two vertices, as Figure a, b, and c illustrate, is an example of trianquads or not.2

2. If you assume that the trianquads are a triangle and quadrilateral connected at a vertex, then Figures a, b, and c would be examples. If you assume that trianquads are a triangle and quadrilateral connected at a single vertex, then Figures a, b, and c would be nonexamples.

3. A teacher would need to include Figure a, b, or c above as either an example or a nonexample of a trianquad. It might also be helpful to include, either as an example or nonexample, a figure that had three common vertices.

Key for Investigation 14.4 (page 14-12)

Activity III: Using Error Analyses

1. a. Roger's drawing implies that all rhombi are rectangles, where as only some are.

b. Rodney's diagram implies that some squares are rectangles but not rhombi and that others are rhombi but not rectangles.

e. Mark's drawing implies that all parallelograms are either a rectangle or a rhombus. Rhomboids (parallelograms like $\square\square\square\square$), however, do not have a right angle or four equal sides.

Key for Investigation 14.6 (pages 14-17 and 14-18)

Part I

Activity 3. A circle could be made to look like an oval in both shadow and balloon geometry. It could be made to look like a straight line in shadow geometry but not balloon geometry. A circle could not made to look like a sidewise 8 in either shadow or balloon geometry.

Part II

As Activity 3 concretely illustrates, only three regular polygons (3-gons, 4-gons, and 6-gons) can tessellate a plane by themselves.

Key for Investigation 14.7 (pages 14-20 and 14-21)

Questions for Reflection

1. A flip is analogous, but not identical to a reflection. A flip of an object changes its position such that its top side becomes its bottom side and vice versa. For example, flipping a coin will change it from heads to tails or vice versa. Now consider a reflection in the mirror. A mirror image shows the same side as the actual object. In the formal discussion of reflection addressed in Question 2 of Activity III, the issue of top and bottom did not come up. In effect, a flip is a three-dimensional model for creating the same result that a reflection in two-dimensional space creates. That is, both a flip and a reflection reverse the orientation of a figure.

2. If you assume that the trianquads are a triangle and quadrilateral connected at a vertex, then Figures a, b, and c would be examples. If you assume that trianquads are a triangle and quadrilateral connected at a single vertex, then Figures a, b, and c would be nonexamples.

3. A teacher would need to include Figure a, b, or c above as either an example or a nonexample of a trianquad. It might also be helpful to include, either as an example or nonexample, a figure that had three common vertices.

2. (a) See page 412 of this guide.

Key for Investigation 14.8 (pages 14-22 and 14-23)

Activity 1

(a) There are limitations on the combinations of rods that can be used to make triangles. That is, a triangle cannot be formed from just any combination of lengths. (b) The sum of any two sides of triangle must be greater than the third side. For example, in a 3, 4, 5 triangle, $3 + 4 > 5$, $3 + 5 > 4$, and $4 + 5 > 3$. In contrast, rods 2, 2, and 5 units long or 2, 3, and 5 units long cannot form a triangle ($2 + 2$ or $2 + 3$ is not greater than 5).
Activity 2

2. Students should discover that the sum of the angles for a 4-sided figure is 360° (twice that for a 3-sided figure), 540° for a 5-sided figure (triple that for a triangle), and 720° for a 6-sided figure (four times that for a triangle). Algebraically, this relationship can be summarized as $S = (n - 2) \cdot 180°$. This equation can also be derived by noticing that a diagonal can divide a 4-sided figure into two triangles. So the sum of its angles should be $2 \cdot 180°$. Two diagonals can divide a 5-sided figure into three triangles; three diagonals can divide a 6-sided figure into four triangles.

Activity 4

See the key for Question 4 of Activity II in Investigation 14.F (on page 427 of this guide).