TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 13•1: Statistics

Many adult students cringe or recoil in pain at the mention of the word statistics. It evokes memories of, for example, completing tedious worksheets that required constructing or interpreting graphs of canned and uninteresting data or droning discussions about the differences among means, medians, and modes. As unappealing as statistics was to learn for many prospective and in-service teachers, the prospect of teaching the topic is even more unbearable (at least for those who are not masochist, sadistic, or both).

A key aim of Unit 13•1 is to help adult students see that, properly taught, statistics can be an engaging topic to teach and learn. Investigation 13.1: How Far and Why? (pages 13-5 and 13-6 of the Student Guide) and Investigation 13.A: Color Effects (page 359 of this guide) can serve to illustrate that statistics instruction should involve collecting, analyzing, and describing real data that address issues, questions, or problems of real concern or interest to children. By actually carrying out one or both of these investigations, students can see how statistics can be integrated with other content instruction (e.g., measurement, operations on decimals, and science) and stem from a rich, worthwhile task.

For Investigation 13.1, most people predict a linear relationship between the height of a tube and the distance a marble will roll. That is, the more the former is increased, the more the latter is increased. They are surprised that the relationship is curvilinear (i.e., increases to a point and then decreases). Why? Intuitively, most students recognize that the higher the tube’s height, the greater the marble’s momentum. However, after an optimal height, the incline becomes so steep that the marble collides more directly with carpet and even bounces, both of which rob it of momentum.

Many students also intuitively predict that a steel marble will roll further than will a glass marble and are surprised this is not always the case. Why? On a carpeted surface, the heavier steel marble sinks further into the carpet and experiences more resistance (friction).

In addition to some interesting science, carrying out the experiment creates a real need to collect data, discuss why multiple trials and finding an average is better than using a single trial (a measurement anomaly has less effect), and to graph data. Instructors may also wish to point out that the experiment creates real opportunities for measuring height and distances, discussing the advantages of using a metric measure (centimeter rulers have finer gradations than do typical inch rulers, metric readings are easily represented as decimals, and decimals are easier to operate on than are fractions), and practicing decimal addition and division (done to compute averages).

Investigation 13.2: Picturing Relationship (page 13-8 of the Student Guide) can highlight the importance of fostering a graph sense and focusing on interpreting graphs rather than on the mechanics of constructing them. (Investigation 13.B: Two Views of a Trip on page 360 of this guide can also help achieve these aims.) Our experience is that Activity I of Investigation 13.2 is moderately difficult for most eighth-grade and college-level students—difficult enough to be challenging, but not so difficult as to be overwhelming. See page 387 of this manual for an interpretation of the graphs in this activity. Activity II can be an interesting way to help students see that graphs reflect real relationships.

Probe 13.1: Using an Everyday Classroom Situation as a Basis for Investigating Averages
and Spread (pages 13-9 and 13-10 of the Student Guide) can be used to illustrate the investigative approach to statistics instruction. Part I can help students understand the underlying conceptional basis for the algorithm for computing a mean. A mean can be viewed in terms of a fair-sharing analogy: What each person would get if their scores were evenly distributed. To do this, the scores first have to be compiled (this is why the first step of the algorithm is to add all the scores) and then divvied up fairly among all the individuals (this is why the second step of the algorithm involves dividing by the number of scores). By summarizing their concrete averaging procedure as a symbolic algorithm, children can be helped to reinvent the algorithm for computing a mean.

Part II can help students see why there are different types of averages. It should make clear that an outlier can so skew a mean that it no longer accurately reflects (i.e., is representative of) the vast majority of scores. In such cases, a median or mode is a more appropriate average. Part III can be used to help underscore the point that an average by itself can be misleading and that reporting a measure of spread also can provide a more complete and accurate picture of a set of data.

Investigation 13.3: Investigating Relationships (pages 13-11 and 13-12 of the Student Guide), Investigation 13.4: Sampling (page 13-14 of the Student Guide), and Investigation 13.5: The Secret Code (pages 13-16 and 13-17 of the Student Guide) illustrate that inferential statistics, as well as descriptive statistics, are appropriate for elementary-level instruction. The first reader inquiry, for instance, can lead to a discussion of how graphs (scatterplots) reflect the strength of a linear relationship (the correlation) between two variables and, in the case of a reasonably strong correlation, can be used to predict an unknown score for one variable from the known score of another variable. Investigation 13.3 can further illustrate the investigative approach, because it begins with a worthwhile task (an interesting problem) and purposefully involves data collection and analysis (including graphing) and other mathematical content, namely, measurement.

Unit 13•2: Probability

For many adult students, the mention of probability elicits about as much enthusiasm as does the mention of statistics. Presented a probability problem, they quickly try to apply, often unsuccessfully, formal prescriptions learned by rote. Presented somewhat novel problems, they misapply their math magic, uncritically accept ridiculous solutions, or throw up their hands in utter despair. The aims of Unit 13•2 include helping readers see that probability can be an engaging topic to teach and learn, that a good many probability problems can be solved informally, and that instruction needs to encourage the use of qualitative reasoning to gauge whether solutions are sensible or not.

Investigation 13.6: Fair Games? (page 13-20 of the Student Guide) illustrates how evaluating the fairness of a game can serve to introduce probability. Students (of any age) can do so by actually playing the games, collecting data on the results, and drawing a conclusion about their results (i.e., by determining each player’s empirical probability of winning). It is also possible to determine the fairness of the games by informally determining theoretical probabilities. Some informal methods might include making a list of possible outcomes, making a table, or using a tree diagram. Investigation 13.D: What are the Chances? (pages 371 to 373 of this guide) can also be used to achieve the aims outlined in this paragraph. Investigation 13.E: Using an Area Model to Informally Solve Probability Problems on pages 374 and 375 of this guide illustrate yet another informal method for determining theoretical probabilities.

Investigation 13.6 can serve to underscore a common difficulty among elementary-level children and even adult students, namely, not recognizing that different outcomes may not be equally likely. For example, in Game 1, children frequently assume that when flipping two coins, heads-heads, a head and a tail, and tails-tails are all equally probable. Likewise, in Game 2, they often assume that the possible outcomes of rolling two dice (the sums 2 to 12) are each equally likely.

Investigation 13.7: Some Problems Involving Probability (page 13-21 of the Student Guide) can help highlight the difficulty of blindly using formal prescriptions to solve probability problems and using the “key-word” approach to choose a procedure. It can further highlight the value of using qualitative reasoning to gauge whether a solution makes sense. These issues are discussed in detail on pages 388 and 389 of this manual.

Note that Problem 2 of Investigation 13.7 often elicits the common error of not considering all
possible outcomes when determining a theoretical probability. Many students assume that because Monique won the coin flip, there is no need to consider outcomes where Unique does so. As a result, their solution to Part a is \( \frac{2}{4} \) (because there are 2 special marbles and a total of 4 marbles), not \( \frac{2}{8} \) or \( \frac{1}{4} \) as the figure below shows. (Some of these students may be mechanically overapplying what they learned about conditional probability.)

In this experiment, coin flip outcomes are: M wins, S* picks special marble, S picks regular marble, N wins, U picks special marble, N picks regular marble.

Investigation 13.8: More "At Least" Problems (page 13-25) can further show how playing games and solving problems can serve as a basis for probability instruction and why using informal methods can be invaluable. Although "at least" problems and how to solve them are typically not introduced in the elementary-grades, such problems can be solved by determining an empirical probability or using informal methods to determine a theoretical probability.

Investigation 13.9: Systematic Simulations of Everyday Situations (page 13-26 of the Student Guide) described how students can create a simulation of everyday events to estimate the relative frequencies of different outcomes. This can serve as the basis for discussing why simulations or mathematical models are an important tool in many fields of human endeavor.

SAMPLE LESSON PLANS

Instructors may choose to spend two or more lessons on statistics and probability for a number of reasons: (a) These topics are key aspects of many everyday situations. (b) They provide a rich opportunity to explore and practice many aspects of mathematics. (c) Many adult students dislike or are even intimidated by these topics and need the opportunity to see that they can be enjoyable to teach and learn. (d) Many adult students have had little or poor statistics and probability instruction in the past. As a result, they need an opportunity to develop their statistics and probability sense and knowledge. The project-based suggestions below may require several classes, weeks, or even months. Although they clearly involve less time than ideal for adequately fostering statistics and probability sense, the single-activities and multiple-activities approaches illustrated below are intended for a two-lesson sequence.

Project-Based Approach

Statistics. One possible project-based statistics unit is exploring the psychological effects of color. See the description of Investigation 13.A: Color Effects (pages 357 and 358 of this guide) and this reader inquiry itself (page 359 of this guide) for specific ideas.

Probability. Another possible project-based probability unit is to devise a game simulation. See Investigation 13.9: Systematic Simulations of Everyday Situations (page 13-26 of the Student Guide) for guidelines in creating a simulation. See Box 13.2: A League of Their Own (pages 13-27 and 13-28 of the Student Guide) for concrete suggestions about devising a baseball simulation.

Statistics and/or Probability. Yet another possibility is to have students, working individually, in pairs, or in small groups of about four, choose their own project. The SUGGESTED ACTIVITIES (on pages 377 to 379 of this guide could serve as a menu. Note that choices 1 and 2 involve students in considering connections between statistics or probability and real-world situations. Choices 2, 3, 4, 7, 8, 9, 14, 15, and 16 involve students in collecting an analyzing their own real data. Choices 4 to 13 require students to apply chapter 13 contents by, for example, analyzing statistics used in real situations or evaluating instructional materials or instruction. Choices 7, 8, 9, and 11 involve them in developing lesson plans or actually working with children. Choices 10, 11, 12, 13, and 14 involve them in examining or analyzing instructional resources. Choices 1, 10, and 12 to 16 involve students in using technology. Note that Choices 3, 14, 15, and 16, in particular, involve both statistics and probability.

Single-Activity Approach

The following suggestions are but a few of many possibilities.

Lesson 2: Probability. To examine empirical probability and informal theoretical probability, an entire lesson could be built around Investigation 13.D: What are the Chances? (pages 371 to 373 of this guide).

Multiple-Activities Approach

Lesson 1: Statistics. For relatively broad coverage of statistics, one possible sequence of activities is:

1. To underscore the need for a new approach to statistics, an instructor can ask adult students about how they were taught statistics, what their attitude toward the topic is, and how much they really understand about it (e.g., Why would you ever use a median rather than a mean? Why would you use a line graph rather than a bar graph?) To illustrate the investigative approach (collecting, analyzing, and describing real data that addresses a genuine issue or question), an instructor can involve them in a science lesson such as the Toothpick-Bug Hunt (Activity File 13.1 on page 13-4 of the Student Guide), Investigation 13.1: How Far and Why? (pages 13-5 and 13-6 of the Student Guide), Activity 1 of Investigation 13.4: Sampling (page 13-14 of the Student Guide), or Investigation 13.5: The Secret Code (pages 13-16 to 13-18 of the Student Guide). (The last is a particularly rich task because it involves both descriptive and inferential statistics.) Even adult students find these activities interesting, which in itself is an important lesson about statistics instruction.

To begin Investigation 13.1, for instance, an instructor can elicit from the class hypotheses about what factors may affect how far a sphere rolls after rolling down an incline. Typical suggestions include the height of the incline, the weight of the sphere, the surface the sphere will roll over, and the initial force with which the sphere was propelled down the incline. Next, the instructor can ask the class how an experiment could be set up to test the first factor, the height of the incline. After discussing the need to control other possible factors, the instructor can explain the steps of the experiment delineated on page 13-5. Next, the instructor can ask students to record their predictions in column 2 of the table on page 13-6 and, then, to represent their predictions graphically in Graph A (on the same page). Note that some students will need guidance filling out the table, and even more may need help translating these predictions into a graph.

After discussing the advantages of using a metric measure and multiple trials, an instructor can ask the class what a mean intuitively means. This can serve as a basis discussing Part I of Probe 13.1: Using an Everyday Classroom Situation as a Basis for Investigating Averages and Spread (page 13-9 of the Student Guide).

After students collect, average, and graph their data and discuss, what for many, are the counterintuitive results, an instructor can prompt reflection about graphs by asking, for instance, “Would it make sense to connect the dots in this graph with straight lines?” This can serve as a basis for discussing the relationship between types of graphs and types of data (summarized below).

Data

Names:
Categorical data
(e.g., gender, favorite color)

Numbers:
Numerical data

Discrete quantities
(collections such as the number of people in class)

Continuous quantities
(measures, such as length)

In this case, a line graph makes sense because both height of the tube and distance of roll are continuous quantities and have “in-betweens” (represented by the points on a line between the plotted points). This discussion sometimes raises questions, such as How can numbers be used to label categories? With any luck, some students will recognize that numbers used in this way represent a nominal meaning, where numbers are essentially used as a name or a substitute for a name (e.g., 0 = male, 1 = female).

2. Probe 13.1: Using an Everyday Classroom Situation as a Basis for Investigating Averages
and Spread (pages 13-9 and 13-10 of the Student Guide) can serve as a basis for discussing key descriptive statistics. For Part I, an instructor may wish to actually have the class play Names-for-a-Number Game. This may raise questions such as, "If a group of six people comes up with 18 unique names for a number, while a group of three people comes up with 15, is it fair that the former be declared the winner? If students don't raise this fairness themselves, an instructor can always play devil's advocate. Students can then be led to recognize that intuitively a mean represents the typical score—the score if each person shared the total scores fairly or evenly. This analogy can help them see why the algorithm for calculating a mean makes sense.

Part II can help students recognize that when a distribution of scores is not even, a mean is not appropriate and, thus, a median or mode should be used. This is an important insight, because statistics instruction typically does not focus on the whys and teachers who understand the rationale for different averages will be more likely to teach this content in a meaningful fashion. To further this discussion, an instructor may ask, "Which average would be appropriate for a bimodal distribution?" (Two modes should be reported in order to represent the data accurately.)

3. Investigation 13.3: Investigating Relationships (pages 13-11 and 13-12 of the Student Guide) can serve to model the investigative approach to statistics and measurement instruction and help adult students recognize that graphs can be useful in making predictions. After discussing the problem, collecting the data, and representing the results as scatterplots, considering the Questions for Reflection at the bottom page 13-12 can help students interpret their graphs. If measurements are taken carefully, a strong relationship should be evident between arm span and height. An instructor may wish to highlight this by placing a pencil over the plotted points to show that they nearly form a straight line. Other body parts will have, at best, only a modest correlation with height. Head size may have a nearly linear relationship but the line will be (nearly) vertical because of the restricted range of head size in most classes. Students should recognize that a vertical line would not be helpful in predicting a height, because any given head size is probably associated with short, medium, and tall height scores. Another issue that frequently comes up is the effect of scale on the readability of a graph. If the range of a scale is too large, then all the data points will be grouped into a very small area, making it hard to determine what kind of relationship exits. By having students regraph the data with a finer scale, they will be able to see that this spreads the data points out more, making the graph easier to read.

SAMPLE HOMEWORK ASSIGNMENTS

Lesson 1
Read: Unit 13•1 of chapter 13 of Student Guide.

Study Group:

- Questions to Check Understanding: 1, 7, 9, and 13a to 13l (pages 379 to 381).
- Writing or Journal Assignments: 1 and 5 (pages 382 and 383).
- Problem: A Dearth of Stopwatches (page 383).
- Bonus Problem: A Passing Grade (pages 383 and 384).

Lesson 2
Read: Unit 13•2 of chapter 13.

Study Group:

- Questions to Check Understanding: 10, 12, and 13j to 13o (pages 380 and 381).
- Bonus Problem: Contestant’s Choice (page 386).

Individual Journals: Writing or Journal Assignment 3 (page 383).

FOR FURTHER EXPLORATION

ADDITIONAL READER INQUIRIES

Investigation 13.A (page 359)

Color Effects illustrates the investigative approach to statistics instruction. This unit explores the psychological effects of color and can provide students with a real reason for collecting, analyzing, and describing data. After completing their
research and drawing their conclusions, students can be encouraged to research the topic and compare their findings with those of psychologists or with the opinions of "experts" (e.g., see the answer key for Investigation 13.A on pages 389 and 390 of this guide, Color: Natural Palettes for Painted Rooms written by Donald Kaufman and Taffy Dahl and published in 1992 by Clarkson Potter, or Color Palettes: From the Donald Kaufman Color Collection written by Suzanne Butterfield and published in 1998 by Clarkson Potter).

**Investigation 13.B** (page 360)

Like Investigation 13.2 on page 13-8 of the Student Guide, Two Views of a Trip illustrates the instructional tip that instruction should focus on interpreting graphs rather than on the mechanics of constructing graphs (page 13-7 of the Student Guide).

**Probe 13.A** (pages 361 to 366)

Analyzing Graphs illustrates various ways of displaying data graphically. Part I involves students in constructing stem-and-leaf plots, a type of graph that may be unfamiliar to many. Part II involves choosing an appropriate graph and can serve as the basis for discussing the relative advantages and disadvantages of various types of graphs. Part III can provide a basis for discussing possible misuses of graphs. For example, the overall shape of a graph can be misleading and that it is essential to interpret graph data with respect to the scale used. (See pages 390-392 for detailed explanations.)

**Investigation 13.C** (pages 367 to 370)

Exploring Averages and Measures of Spread can serve as a basis for helping adult students understand the rationale for and meaning of these two basic descriptive statistics. See pages 393 and 394 of this guide for a more complete discussion.

**Investigation 13.D** (pages 371 to 373)

What are the Chances? entails analyzing a common game of chance and, thus, provides a worthwhile task for exploring several aspects of probability. In discussing either empirical or theoretical probability, students need to be able to recognize outcomes—possible results. For this activity, for instance, they need to recognize that a one on a green die and a two on a red die is one outcome and that a two on the green die and a one on the red die is a different outcome. Moreover, they need to identify the sample space—all the possible outcomes. For this activity, for instance, the sample space is the 36 possible sums of two dice. An event is one outcome or some combination of outcomes in the sample space. For example, in this activity, there are three possible events: reroll the dice (rolling a 4, 5, 6, 8, 9, or 10), an automatic win (rolling a 7 or 11), and an automatic loss (rolling a 2, 3, or 12).

**Investigation 13.E** (pages 374 and 375)

Using an Area Model to Informally Solve Probability Problems illustrates how an area analogy and Fraction Tiles can be used to solve a variety of probability problems.

**QUESTIONS TO CONSIDER**

1. Babe Ruth hit 22, 25, 34, 35, 41, 41, 46, 46, 46, 47, 49, 54, 54, 59, and 60 homeruns in his career as a New York Yankee. In 1961, Roger Maris broke Ruth’s record by hitting 61 home runs. In his six other years as a Yankee, Maris hit 8, 13, 23, 26, 33, and 39 homeruns. (a) Construct a stem-and-leaf plot to compare the two hitters. (b) Who was the superior homerun hitter and why?

2. Circle the letter of any statement below that, according to this guide, is true. Change the underlined portion of any false statement to make it true.

   a. Graphing should be introduced with bar graphs.

   b. A line graph is appropriate for showing the relationship between time and volume of water in a pail as it is being filled.

   c. A line graph is not appropriate for showing the relationship between the 31 flavors of ice cream and the number of people who favor each.

   d. A picture bar graph is appropriate for illustrating categorical data (e.g., the number of consumers who preferred BLAST-GAS, TUMMY-MOMMY, or ACIDAWAY antacid pills). (Continued on page 376.)
Investigation 13.A: Color Effects†

◆ Collecting, graphing, and analyzing data ◆ K-8 ◆ Group or class project

Color is regularly used in songs, literature, and paintings to describe or portray feelings. But do colors really have any effect on how we feel? Can the colors we wear affect our mood? Can the color of a room, the color of a soap box, or the color of a political poster affect how we feel? Apparently, there is a physiological basis for thinking so. Color stimulates nerve cells in our eye called cones, which initiate signals to the brain. Some of these signals are routed to the pituitary gland, which helps control how we feel. As an integrated literature-science-mathematics lesson, encourage students to evaluate the effects of color on feelings. The complexity of the investigation can vary with grade level. To see what is involved and to explore the issues raised above, try the activity yourself, either on your own or with your group. Share your findings with your group or class.

Possible Activities

Survey: Color Preference. With kindergarten children, a teacher may simply pose the question, “What is your favorite color?” The children could then be given a chance to choose from differently colored rectangles cut from construction paper. After printing their name on the rectangles, children could then place them on a bulletin board above the label for their favorite color. Note that this bulletin board also creates a need for children to read color words and associate them with the correct color.

Survey: Effects on Feelings. A class or group of students can conduct a survey of their own class, grade level, or the school. Particularly with younger children, a teacher may wish to begin by suggesting questions such as the following: (1) What color makes you feel calm? (2) What color makes you feel worried? (3) What color makes you feel safe?

Survey: Real-World Applications. Older students, particularly, may wish to consider real-world applications of “color science” such as: (1) What suit color might help a politician or business person command respect? (2) What box color might help a detergent manufacturer create the impression its laundry soap is powerful? (3) What color should a counselor wear to inspire trust? (4) What color should a person wear to create the impression of goodness? (5) What color can help create the impression of power? (6) What color should ice cream manufacturers avoid? Students may wish to add or substitute their own questions. It may be less confusing to have younger students pick from a limited list of colors. Older students can be given a wider list of choices and even the option of citing an unlisted color.

Science Experiment. Another option is for students to conduct their own color-effect experiments. In a quiet place, they can have participants stare at color samples. After looking at yellow, say, for one minute, have a participant fill out a questionnaire such as: Does this color create a feeling of (a) calm, (b) anxiety, (c) happiness, (d) sadness, (e) none of the above. Older students might wish to experiment with a (Likert-like) scale rating colors on several dichotomies such as:

<table>
<thead>
<tr>
<th>Very calm</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Very anxious</th>
</tr>
</thead>
</table>

Science-Experiment Extension. With older students, particularly, a teacher may wish to discuss why colors may stir specific feelings. Is there an innate basis, or are color associations learned? In other words, might different individual and cultural experiences affect a color’s impact? What social implications do your results raise?

Possible Analyses

After students have collected their survey or experimental data, have them consider how they can represent it to facilitate analysis (e.g., summarizing trends). Note that creating graphs would be an ideal tool for such a task.

Compare the class survey and experimental results with the data listed on pages 389 and 390 of this guide. In what ways does the class’ results agree with these findings? What might account for discrepancies? Is there any truth to the conjecture that color can have an effect on feelings?

† This activity was inspired by an article (“Color: There’s a Lot More to Color Than Meets the Eye” authored by Deborah Heiligram) appearing in the November 1991 issue (pp. 16-19) of 3•2•1 CONTACT, a children’s magazine published by Children’s Television Workshop.
Investigation 13.B: Two Views of a Trip

- Interpreting graphs  
- 7-8  
- Small groups of four + class discussion

The following activity, inspired by a lesson developed by the Shell Centre for Mathematical Education (1985), requires students to interpret two different graphs (perspectives) of an everyday situation. With your group, try it; then share your results with your class.

The Jorgensons decided to go to the auto races in Indianapolis, a drive of 115 miles from their home in Urbana. A map and a graph of their automobile trip is shown below:

1. Using the map and graph above, describe what happened between Points A and B, B and C, C and D, D and E, E and F, and F and G. For each stage of the trip, include the location.

2. On a piece of graph paper, graph the car’s speed in m.p.h. as it goes from point to point over time (in hours). Make time the horizontal axis.

3. Devise a story about the Jorgensons’ car trip using the information shown in the map and the graphs above. Your story should explain what happened at each stage of the trip (e.g., what happened between Points A and B).

The aim of this probe is to familiarize students with some of the many ways to organize data graphically. After examining the examples below, answer the questions in Part I (pages 363 and 364), II (pages 364 and 365), and III (pages 365 and 366) and the Questions for Reflection (page 366).

Some Types of Graphs

The type of graph chosen depends partly on whether categorical or numerical data were collected. Bar graphs and their cousins (Graphs A to E below) are useful for illustrating categorical data—the total number (frequency) for each category. Some variables such as gender involve two categories (e.g., the number of boys and girls in a class), and some such as color involve more than two categories (e.g., the number of children choosing red, orange, yellow, green, blue, and so forth as their favorite color). Note that categories involve names rather than numbers and, thus, do not have any inherent numerical order or magnitude. For example, it does not make sense to say that red is larger than blue. Other graphs are useful for illustrating the frequency or measurement of numerical data—of variables that have a numerical order (e.g., the number of children with successively higher percentile scores). Graphs F, G, and H below are useful for illustrating numerical data consisting of at least one discrete quantity. Graph I below is useful for illustrating numerical data comparing two measurements or continuous quantities such as temperature over time. Circle graphs (Graph J) are in a class of their own because they represent ratios.

A. Real graph*

Real graphs use real objects (e.g., in Example A, the children themselves). Placing the objects on a grid can make comparisons easier.

B. Picture graph*

Birthdays Each Month

<table>
<thead>
<tr>
<th>Month</th>
<th>Bill</th>
<th>Jane</th>
<th>Dick</th>
</tr>
</thead>
<tbody>
<tr>
<td>September</td>
<td><img src="example.png" alt="Bill" /></td>
<td><img src="example.png" alt="Jane" /></td>
<td><img src="example.png" alt="Dick" /></td>
</tr>
<tr>
<td>October</td>
<td><img src="example.png" alt="Oscar" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>November</td>
<td><img src="example.png" alt="Edie" /></td>
<td><img src="example.png" alt="Jo" /></td>
<td></td>
</tr>
</tbody>
</table>

Picture graphs (pictographs) use pictures or icons (images).

D. Symbol graph (using Xs)

Birthdays Each Month

<table>
<thead>
<tr>
<th>Month</th>
<th>Xs</th>
</tr>
</thead>
<tbody>
<tr>
<td>September</td>
<td><img src="example.png" alt="Xs" /></td>
</tr>
<tr>
<td>October</td>
<td><img src="example.png" alt="X" /></td>
</tr>
<tr>
<td>November</td>
<td><img src="example.png" alt="Xs" /></td>
</tr>
</tbody>
</table>

Symbol graphs use geometric shapes such as □ or ○, Xs, or some other symbol to represent items rather than pictures or icons.

*These graphs can run vertically, as shown, or horizontally.
Probes 13.A continued

Some Types of Graphs continued

E. Bar Graph

Bar graphs use rectangular bars, which can run either vertically, as shown, or horizontally.

F. Histogram

Histograms represent grouped data (e.g., in Example F, note that the heights are listed in ranges of 3 inches). In effect, they are a bar graph where consecutive categories form an ordered numerical scale.

G. Line Plots

Line graphs represent each piece of data along a numerically ordered scale.

H. Stem-and-Leaf Graphs

With stem-and-leaf graphs, the numerals of the data are plotted. In Example H, the tens digits form the stem and the ones digits form the leaves. Part I below discusses this type of graph in more detail.
I. Line Graph

Line graphs are useful for representing the relationship between two continuous variables—two sets of data where it makes sense to have in-betweens. In Example I, for example, a line connecting two plotted points makes sense because both time and depth can be considered continuous quantities. Time can be viewed as a succession of days and a fractional part of day makes sense. Depth likewise forms a numerically ordered scale, which can involve fractional parts.

J. Circle Graph

Circle graphs or pie charts are used to present how a whole is divided among its parts and, thus, to compare the relative size of the parts. Unlike the other graphs depicted, then, this type of graph represents ratios rather than absolute values.

Part I: Stem-and-Leaf Plots

Stem-and-leaf plots are a relatively new type of bar graph. Even if you are familiar with this increasingly popular way of organizing data, you may wish to check your knowledge by answering the questions below.

1. Mr. Lemon’s class had the following scores on a mathematics achievement test:

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>84, 23, 9, 45, 36, 75, 7, 18, 28, 42, 50, 64, 12, 25, 45, 52, 44, 61, 48, 93, 74, 40, 33, 62</td>
</tr>
</tbody>
</table>

Because the scores were listed in the alphabetical order of his students’ name, he had trouble making much sense of the data. To help him organize the data, Mr. Lemon decided to make a stem-and-leaf plot—using the tens digit of the scores as the stem and the ones digit as the leaf (see Table 1). To further organize the data, he decided to redo the plot so that the leaf entries were in numerical order. Complete Table 2 below listing the leaf entries from left to right in increasing size.

Table 1: Stem-and-Leaf Plot (Unordered listed)  Table 2: Stem-and-Leaf Plot (Increasing order)

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9 7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>8 2</td>
<td>2</td>
<td>3 8 5</td>
</tr>
<tr>
<td>2</td>
<td>6 3</td>
<td>3</td>
<td>2 5 4 8 0</td>
</tr>
<tr>
<td>4</td>
<td>5 2 5 4 8 0</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>0 2</td>
<td>6</td>
<td>4 1 2</td>
</tr>
<tr>
<td>7</td>
<td>5 4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

2. Mrs. Orange’s class had the following achievement scores: 92, 85, 77, 56, 66, 95, 99, 93, 70, 80, 82, 76, 50, 62, 48, 45, 75, 41, 85, 87, 72, 60, 67, 54. To group her scores by five, Mrs. Orange constructed Table 3 below. The first three scores have already been entered into the table. Complete the table.

Table 3: Stem-and-Leaf Plot (Increasing Order, Grouped by Fives)

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>
Probe 13.A continued

3. Mr. Lemon wanted to compare his class’ scores with those of Mrs. Orange’s class. How could he add her data to his to make such a comparison visually apparent? Illustrate your solution by modifying Table 4 below and adding the necessary data.

Table 4: Stem-and-Leaf Plot for Comparing Two Sets of Data

<table>
<thead>
<tr>
<th>Mr. Lemon’s class</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

4. Marsha, who had been competing in a national science contest returned to Mr. Lemon’s class after a month’s absence. After taking a make-up test, her mathematics achievement score was 100. How could Table 4 be modified to take into account this new score?

Part II: Choosing an Appropriate Graph

Learning how to use graphs involves the ability to choose an appropriate graphic representation for a situation. This requires understanding the advantages and limitations of each type of graph. These issues can be raised by an activity such as the one described below.

1. To help determine the class’ favorite dessert, Ms. Fair-Field asked her students how they could record and organize their data. One suggestion was create a graph. After a discussion of how some items should be classified, Ms. Fair-Field asked what type of graph would be appropriate to determine the total (frequency) of each type of dessert. (a) Would a real, picture, picture bar, or symbol graph be appropriate for representing the frequency of each type of dessert? Why or why not? (b) Would a bar graph? Why or why not? (c) Would a line plot, or a histogram? Why or why not? (d) Would a circle graph? Why or why not?

2. As each child announced his or her favorite dessert, each group in Ms. Fairfield’s class recorded the data. Reyniel’s group summarized this information in the graph below. (a) Is a line graph an appropriate way to represent these data? Why or why not? (b) Under what circumstances are line graphs appropriate? Illustrate your conclusion with an example.

3. If Ms. Fair-Field’s class wanted to depict the favorite dessert of the whole school, which type of graph would be appropriate?

4. Ms. Fair-Field’s class used package information and food charts to gauge the total grams of fat in each lunch. (a) What type of graph would be appropriate for representing how many lunches had various amounts of fat? (b) To make the graph more compact, Ms. Fair-Field suggested using levels of fat grams—0-5 grams, 6-10 grams, 11-15 grams, and so forth. What type of graph would be appropriate?

5. The class next analyzed the lunches in terms of how many included no fruit or vegetable, a fruit but no vegetable, a vegetable but no fruit, both a fruit and a vegetable. (a) What graphs would be appropriate for picturing these results? (b) If the class wanted to represent the percent of lunches that fell into each category, what type of graph would be appropriate?

6. The class wanted to compare the frequency of their use of fruit and vegetables to that of Mr. Randolph’s class next door. This neighboring class had the same number of students. (a) Could the graphs cited as answers to Question 4a be modified so that a comparison of the two classes was possible? If so, how? If not, why not? (b) How could the students compare the two classes if percents were used?
Probe 13.A continued

7. The class also wanted to compare the frequency of their use of fruit and vegetables to that of Mrs. Chavez' other neighbor. This class, however, had three more students. Which type of graph would best facilitate readily making a fair comparison? Justify your choice.

8. Mrs. Chavez's class gauged the fat content of three examples from each of six food groups (fats, oils, & sweets; milk, yogurt & cheese; meat, poultry, fish & eggs; vegetables; fruits; and bread, cereal, rice, and pasta). To get a sense of what food group had the largest and smallest fat content, they grouped the data as shown below. Which of the following graphs would provide a good picture of what food group tended to be high in fat and what tended to be low: picture graph, bar graph, histogram, or circle graph?

- **FATS, OILS, SWEETS**
  - butter, olive oil, chocolate bar
- **MEAT, POULTRY, FISH, & EGGS**
  - hamburger, fish fillet, egg
- **FRUITS**
  - apple, orange, banana
- **MILK, YOGURT, & CHEESE**
  - whole milk, yogurt, American cheese
- **VEGETABLES**
  - lettuce, carrot, celery
- **BREAD, CEREAL, RICE, & PASTA**
  - white bread, Cheerios, spaghetti

9. To see if there were any trends by grade in the consumption of fruit, Mrs. Chavez's class randomly surveyed 20 children about how many servings from the fruit group they ate a day. They graphed the mean response by grade level. Which of the following graphs would help the class to see any trends by grade level: picture graph, bar graph, histogram, or circle graph?

**Part III: Possible Misuses**

Learning how to use graphs also involves understanding how graphs can be misused or be misleading.

1. Mr. Sneaky Snead wanted to be promoted to CEO of Gross National Products. To make his case that he, not the other senior vice president, should be promoted, Sneaky devised the graphs below for the board of directors. (a) Particularly for the inattentive board member suffering from ME-GLO (my eyes glaze over), which graph seems to convey the larger growth in productivity? (b) Is a superficial impression substantiated by the facts? Why or why not? (c) Why do the graphs create the impression they do?

![Productivity of Mildred's Eastern Division Since Taking Over](image1)

![Productivity of Sneaky's Western Division Since Taking Over](image2)
3. Sneaky further noted that current sales in Mildred’s division were 40,000 units and projections for next year’s sales were for twice that. “Compare that with the current and projected sales for my division,” he argued, showing the graph below. What impression is Sneaky trying to create and how does he do it?

4. Next, Sneaky explained that Mildred spent 50% of her workday supervising subordinates, 20% in conferences, 12\(\frac{1}{2}\)% at lunch, 10% on paperwork, 5% on customer relations, and 2\(\frac{1}{2}\)% on other. Showing the graph below, Sneaky added, “This graph clearly shows my superior use of work time.” How might his graph be misleading?

Questions for Reflection

1. Miss Brill had her class construct a picture graph to help them decide which game to play for P.E. Children were invited to tack a card with their names next to their choice. Unfortunately, this was done in a haphazard manner, making it difficult to tell at a glance which game had the most votes. Indeed, in several cases, it was unclear what game the voter had intended. How could Miss Brill’s class make their voting mechanism more readable in the future?

2. (a) What type of graph would be useful in representing achievement gains of 80%? (b) Could the same graph be used to represent achievement gains of 150%? Why or why not?

3. To represent the relationship between two continuous quantities, such as the length of a meter stick’s shadow over the course of a day (time), the weight of a cart versus the distance the cart travels after traveling down an incline, or the outside temperature over the course of a day (time), what type of graph would most precisely represent the data?

4. To represent that he spent 33% of his day sleeping, Alejandro drew the circle graph to the right. Evaluate his effort.
Although most people equate average with a mean score (the arithmetic mean), an average can also be expressed as a median (the middle or 50th percentile score) or a mode (the most frequent score). The mean, median, and mode of 1, 3, 3, 4, 8, 11, for example, is 5, 3.5, and 3, respectively. Spread indicates how a set of data is dispersed or spread out. A relatively simple way of indicating the spread is to report the range—the difference between the smallest and largest value (e.g., the range of 1, 3, 3, 4, 8, 11 is 11 - 1 or 10).

To see how the aims listed at the top of the page are achieved and to perhaps deepen your own understanding of averages and spread, try the following activities.

Activity I: Some Average Cases (3-8 Reasons for averages)

Solve the following problems and then answer the questions that follow. Discuss your answers with your group or class.

- **Problem 1: A Fair Grade.** Janna had quiz grades of 10, 9, 10, 8, 7, 10, and 9. If 10 = A+, 9 = A, 8 = B, and 7 = C, what letter grade would fairly represent Janna's overall performance on the seven quizzes?

- **Problem 2: Comparing Batting Prowess.** In the first four games, Bud had 7 hits and C. J. had 6. Bud tried to claim he was the better hitter, but C. J. pointed out that as the lead batter, Bud had more chances to get a hit—20 to his 16. Who is the better batter?

- **Problem 3: Predicting Earning Power.** Tabitha needed $10 for a class trip and was completely broke. Over the past 6 weekends she had earned $6, $16, $8, $15, $12, and $9 babysitting. What could she expect to make on the next weekend?

1. Why are averages useful?
2. Problems 1, 2, and 3 above could be solved by computing a mean. Intuitively, what is a mean?

Activity II: Concretely Modeling the Mean with a Leveling Analogy (3-8 Informally representing the mean)

1. If each score is thought of as an elevation, what would the mean represent? Use scores of 5, 3, 2, 1, 1, 1 as an example.

2. The mean can be viewed as the center point of the elevations. This center point can be determined by "leveling out" the elevations representing each score—by rearranging the heights of the individual scores so that they all have the same elevation. Illustrate with tiles or with a diagram how a leveling method could be used to determine the mean of the scores: 5, 3, 2, 1, 1, 1, and 1.

3. Illustrate how the task above could be revised so that students might recognize that the mean average can be a hypothetical score rather than one of the scores in the set.

4. In the leveling analogy, the total number of blocks taken from higher elevations equals the number of blocks added to lower elevations. What point about a mean might this help students to see?

5. (a) Illustrate how a leveling analogy could be used to find the mean of the following set of scores: 5, 4, and 3. (b) Would the mean change if a fourth score of 0 was added? Why or why not? Consider how the leveling analogy could be used to justify your answer.

Activity III: A Balance Problem (3-8 Informally representing the mean)

Consider the following problem:
Investigation 13.C continued

- **Problem 4: Balance Point?** Using the unnumbered side of a *Math Balance*, devise a scale and place weights on the scale at 2, 5, 8, 12, and 18 so that the weights balance.

  Hint: What does the fulcrum (balance point) represent and what is its value?

**Activity IV: A Reverse-Process Problem (◆ 5-8 ◆ Exploring the characteristics of a mean)**

Solve the following problem:

- **Problem 5: Grade Worry.** Denny was really concerned about his grade on the last math project. From his desk, he could see that his group’s average was 75 and that the grades of his three group mates were 95, 90, and 80. The grade of his fourth group mate and his own grade were covered by a book. (a) What is the highest grade Denny could possibly get? (b) What is the lowest? (c) If his fourth group mate got his usual grade of 60, what would Denny’s grade have to be?

1. In Problem 5, where is the mean of the group located in comparison to the group member’s grades?
2. If the difference between each group member’s grade and the mean were added up, what would the sum be?
3. Based on Activities II, III, and IV, what characteristics does a mean have?

**Activity V: Informally Reinventing the Algorithm for Determining Means (◆ 5-8 ◆ Connecting-level activity)**

Children can be encouraged to re-invent the formal procedure for computing a mean by building on their fair-sharing knowledge. Consider the following word problem:

- **Problem 6: Don’t Be Mean, Take the Mean.** Mr. Jessup put a basket of candies on the table. Ahmed, the closest, grabbed eight; Beatrix, the quickest, grabbed seven; Chelsey, the largest, grabbed five; Daniella, the undaunted, also grabbed five; Evan, the smallest, grabbed four; and Francois, the slowest, got only one. Responding to the resulting fuss, Mr. Jessup noted calmly but firmly, “Everyone is to get the same number of candies.” How many candies should each child get?

  1. How could blocks be used to solve Problem 6 informally?
  2. Represent your informal procedure as symbolic arithmetic procedures. What arithmetic operations did you use and why?
  3. Determining the size of a fair share as you did in Problem 6 is the same as finding the *mean* (share). If 9, 8, 5, and 2 items were shared fairly among four children, what would each child’s share (the mean) be? Illustrate your solution process with (a) blocks and (b) symbolic arithmetic procedures.

**Activity VI: Scores of Different Weight (◆ Weighted averages ◆ 6-8)**

Solve the following problem and discuss your solution with your group or class.

- **Problem 7: Whose Ahead?** After the first day of an all-class academic olympics, Avita’s group had 5 gold (first-place) medals, 0 silver (second-place) medals, and 1 bronze (third-place) medal. Bootswalla’s group had 1 gold, 5 silver, and 3 bronze. And Craig’s group had 2 gold, 4 silver, and 2 bronze. Avita argued that her group was in first place because they had the most gold medals. Bootswalla argued that his group should be considered the front sprinter because they had the most medals overall. What would be a reasonable way of deciding whose ahead?

**Activity VII: A Problem in Salary Negotiations (◆ Exploring the characteristics and roles of different averages ◆ 6-8)**

- **Problem 8: Salary Impasse.** Your request for a salary hike has been turned down by the school board. They note that the average salary for the 11 elementary teachers in the district is higher than the average salary for elementary teachers with comparable experience, performance (as indexed by the number of state scholarship
Investigation 13.C continued

winners), and load in the surrounding districts. Analyze the board's data below and evaluate the merits of its case. Then answer Questions 1 to 3.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Yrs.</th>
<th>Salary</th>
<th>Scholarship Winners</th>
<th>Class Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40</td>
<td>$42,500</td>
<td>4</td>
<td>22</td>
</tr>
<tr>
<td>B</td>
<td>7</td>
<td>23,000</td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>23,000</td>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>D</td>
<td>7</td>
<td>23,000</td>
<td>3</td>
<td>23</td>
</tr>
<tr>
<td>E</td>
<td>7</td>
<td>23,000</td>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>F</td>
<td>5</td>
<td>22,000</td>
<td>3</td>
<td>26</td>
</tr>
<tr>
<td>G</td>
<td>5</td>
<td>22,000</td>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>H</td>
<td>5</td>
<td>22,000</td>
<td>3</td>
<td>24</td>
</tr>
<tr>
<td>I</td>
<td>3</td>
<td>21,000</td>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td>J</td>
<td>3</td>
<td>21,000</td>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>K</td>
<td>1</td>
<td>20,000</td>
<td>2</td>
<td>24</td>
</tr>
</tbody>
</table>

Average 8.2 $23,864 3 24.6

School District

<table>
<thead>
<tr>
<th>School District</th>
<th>Yrs.</th>
<th>Salary</th>
<th>Scholarship Winners</th>
<th>Class Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8.1</td>
<td>$23,100</td>
<td>3</td>
<td>24.3</td>
</tr>
<tr>
<td>B</td>
<td>8.3</td>
<td>$23,600</td>
<td>4</td>
<td>26.1</td>
</tr>
<tr>
<td>C</td>
<td>8.2</td>
<td>$23,400</td>
<td>3</td>
<td>25.3</td>
</tr>
</tbody>
</table>

1. Examine each graph at the bottom of the page. (a) Does the mean accurately reflect the vast majority of scores in each case? (b) What general conclusion can you draw about when the mean is useful and when it is not?

2. In each case, what is the median? In what cases is it identical to the mean, similar to the mean, and rather different from the mean? When is the use of the median particularly appropriate?

3. Problem 6 of Activity V listed scores of 8, 7, 5, 5, 4, and 1. Would the median be identical, similar, or different from the mean in this case? Would it be more appropriate to use the median than the mean with such a set of scores? Why or why not?

Activity VIII: Choosing the Most Appropriate Average

Analyze the following problem situations. For each, note which average (mean, median, or mode) would be most appropriate. Justify your answer. Share your answers and reasoning with your group or class.

Situation 1. Abdul saw that Ms. Purine had finished grading their term papers that were to be turned back the next day. Impatient to learn how he had done, Abdul fabricated a reason to speak to the teacher. While engaging the teacher in conversation, he noticed a summary of the paper grades on her desk:

Taking into account that Ms. Purine graded on a scale from 0 (F) to 4 (A), Abdul concluded that five papers got an A, seven got a B, six got a C, five got
a D, and 2 got an F. What is the mean grade of the papers? What is the median grade? What is the mode? Which would be most useful in gauging Abdul’s most probable grade?

**Situation 2.** The players on Riccardo’s team had the following head sizes: 7\(\frac{1}{2}\), 5\(\frac{3}{2}\), 7, 7\(\frac{1}{2}\), 7, 7\(\frac{1}{2}\), 7. The sporting good salesman offered him a large discount if he ordered the same-size caps. “Just order the average head size,” the salesman suggested. What is the mean, median, and mode head size of the team and which would be useful in this situation?

**Situation 3.** To prompt a discussion of healthy diets, Miss Fetticini had her class vote on a treat for snack time. She had the class summarize the results in the symbol graph shown below. What type of average (mean, median, or mode) would be useful in describing the typical choice and which would not? Why?

<table>
<thead>
<tr>
<th>Fruit</th>
<th>Popcorn</th>
<th>Candy</th>
<th>Celery &amp; Carrots</th>
<th>Cupcakes</th>
<th>Potato chips</th>
<th>Pretzels</th>
</tr>
</thead>
</table>

**Situation 4.** Fernand got scores of 10, 9.5, 9, 8, and 7.5 from the judges for his skating. Which type of average would make sense in this situation and which does not? Why?

**Activity IX: Cases of Incomplete Data (♦ Concept of spread ♦ 4-8 ♦)**

Consider each of the following situations. Identify those where the information given is incomplete and, thus, misleading. In such cases, identify what additional information is needed to make an informed decision.

**Situation 1.** The mayor of the Gowestyoungman Township advertised to buy a lot straddling the Mithippi River for a community day care center that would have a wading area. Mrs. Valapriso, a real estate agent, offered a lot with the following recommendation: *The average depth of the water through the lot is only 1 foot.* Is this the ideal lot the mayor was looking for?

**Situation 2.** Alecia scored a 600 on her SIT (Scholar Identification Test). She pointed out that such a high score clearly indicated above average scholarship and that such effort surely deserved a small token of appreciation such as a new car. How strong is Alecia’s case really?

**Activity X: Box-and-Whisker Plots (♦ Visually displaying the median, the range, and distribution of the data in terms of quartiles ♦ 5-8)**

The teams of the two sixth-grade classes at Abbie Normal Elementary School had each won the same number of games, and the weather did not permit another game before the district playoffs. The classes agreed that the team with the better overall hitting performance should go to the district playoffs. The number of hits for each player is listed below, and the data from Mr. Arnell’s class is displayed in a box-and-whisker plot.

- **Mr. Arnell’s class:** 5 7 9 10 12 13 15
  - 25 30
- **Mrs. Yackell’s class:** 8 9 11 12 14 15 16
  - 18 20

The plot above was constructed by plotting the lowest and highest score. The ends of the box were constructed by finding the lower quartile and upper quartile. The former is the median of the lower half of scores; the latter, the median of the upper half of score. (In both cases, the overall median is included if there are an odd number of scores). The line in the box represents the overall median.

1. Construct a box-and-whiskers plot for Mrs. Yackell’s class.
2. (a) The box in each graph contains about what fraction of the data? (b) The left-hand and right-hand whiskers each represents about what fraction of the data?
3. (a) Overall, which class has the more consistent batters? Why? (b) The middle half of which class has the more consistent hitters? Why?
Investigation 13.D: What are the Chances?

- Exploring empirical and theoretical probability
- 3-8
- Small groups of about four

*Chancy Dice* (commonly known as *Craps*) is played with two regular dice (six-sided dice with faces of 1 to 6 dots). If a shooter rolls dice summing to 7 or 11 on the first roll, she wins. If the shooter rolls dice summing to 2, 3, or 12 on the first roll, she loses. If any other sum is rolled, then the shooter must reroll this sum before rolling 7 or 11 to win.

This game raises a number of questions that can lead to an investigation of empirical (*experimentally-determined*) probability, theoretical (*logically-deduced*) probability, or both. (With primary-level children, a teacher can focus on investigating empirical probability; with intermediate-level children, a teacher can—depending on the students’ prior experience—explore both empirical and theoretical probability or just the theoretical probability.)

- What are all the possible outcomes of rolling a red die and a green die if both are regular dice?
- On the *first* roll, is a shooter more likely to win (roll a 7 or an 11) or lose (roll a 2, 3, or 12)? Why?
- The question above raises the following issue: Are rolls of 2, 3, 7, 11, and 12 equally likely? For example, does a 12 stand as good, a better, or a worse chance of coming up than does a 2? 3? 7? 11? Why?
- If a 6 comes up on the first roll, is a shooter more, equally, or less likely to win? That is, is she more, equally or less likely to roll a 7 or an 11 on subsequent rolls? Why?

To see what is involved and, perhaps expand your probability sense, try Activity I with your group (or by yourself). Then complete Activities II and III with your class (or group).

**Activity I: Gauging Empirical Probability**

To address the question above, conduct the following experiment:

- Roll two regular dice (one red and one green) and determine the sum.
- In Table A (on page 373), record a tally mark next to the sum.
- Repeat Steps 1 and 2 until you have 36 entries in Table A.
- Determine the empirical probability for each possible sum—the numbers 2 to 12. The empirical probability is:

\[
\text{Number of rolls a sum came up} \quad \frac{\text{Total number of rolls}}{1. \text{ Did each of the sums 2 to 12 occur about equally often?}}
\]

2. Did 12 occur more, the same, or less often than 2? 3? 7? 11?

3. On the first roll of *Chancy Dice*, is a shooter more, equally, or less likely to win than lose?

4. After throwing a 6 on the first try, is a shooter more, equally, or less likely to win (throw a 6) or lose (throw a 7 or an 11)?

5. The empirical probabilities your group calculated are estimates of the chances a sum would be rolled on any given roll. According to the empirical probabilities you have calculated, how many 12s could you expect if the dice were rolled another 36 times? How many 11s? How many 2s? How many 12s could you expect if the dice were rolled another 72 times instead? How many 3s could you expect if you rolled the dice just 18 times instead? How many 7s? How many 12s? How many 3s? (Note that the last three questions involve proportional reasoning.)

6. How could you get a more accurate estimate of the probability of rolling each sum?
Investigation 13.D continued

Activity II: Exploring Empirical Probability Further

Increasing the sample size is a good way to increase the reliability of a measure. We can increase our sample size by combining the results obtained by each of the groups in the class. Enter the totals for each sum 2 to 12 obtained by each group in the appropriate column of Table B on page 373. Calculate the overall total for the class for each sum and enter these figures in the appropriate column of Table B.

1. How could you calculate the empirical probabilities for the class so that you could easily compare it to your group’s empirical probabilities?

2. One way to generate comparable data is to calculate the class’ mean empirical probabilities. \[ \text{It may help to use a calculator. Alternatively, an instructor can have the classroom computer(s) set up so that each group can input the raw data and then use a spread sheet to calculate the average. To practice using decimal addition and division, round off your answers to the nearest tenth or hundredth rather than the nearest whole number.} \] For example, if your class has six groups and the overall total for a sum of 2 is 8, what would the mean rate be? What would the mean number of total rolls be? What would be the mean empirical probability for rolling a 2? Enter the means for each sum in the overall-empirical-probability column of Table B.

3. Which, if any, answers to the questions of Activity I changed when you use the class’ data?

Activity III: Informally Exploring Theoretical Probability

An even more reliable estimate of the probability of rolling each sum could be determined by repeating the experiment many more times. A really good estimate could be determined by repeating the experiment an infinite number of times. But this or even a very large number of trials is not practical. A shortcut for estimating the chances of an event is to analyze the situation logically—to determine the theoretical probability.

A logical analysis can be aided by reflecting on some concrete examples and organizing the data in a chart or a table. Imagine rolling a red die and a green die. The sum 3, for example, could be obtained with a 1 on the red die and a 2 on the green, or a 2 on the red die and a 1 on the green. Thus, there are two ways of getting this sum. (Note that a red 1 and green 2 is not different from green 2 and red 1. The order in which the dice are thrown is irrelevant—it does not change the fact that there is a red 1 and green 2. On the other hand, a red 2 and green 1 represents a different combination.)

1. Using a green and a red pencil or crayon, list all ways to get each sum in Table C on page 373. Using the six-by-six table to the right may aid in this process.

2. For each sum 2 to 12, compute the absolute difference between your group’s empirical probability (Table A on page 373) and the theoretical probability (Table C) and the absolute difference between the class’ (the mean) empirical probability (Table B) and the theoretical probability (Table C). For both your group’s and your classes’ data total the differences. Which set of data was, in fact, closer to the theoretical probabilities, your group’s or the class’ empirical probability?

Questions for Reflection

1. Why it is important to use two differently colored dice when trying to help students figure out all the possible combinations that make a sum?

2. How could the tasks posed by this investigation be made easier for primary-level children if need be?
Investigation 13.E: Using an Area Model to Informally Solve Probability Problems

- Informally solving probability problems
- 5-8
- Any number

As probabilities can be expressed as fractions and fractions can be represented by an area model (e.g., by using Fraction Tiles), it follows that probability problems can be represented by an area model also. To see how, tackle the following problems, preferably with the help of your group. Discuss your solution strategy and solutions with your class.

Problem 1: Foiled Excuse

Two college students who had not studied sufficiently for their final exam in Chemistry concocted a scheme to get more studying time. They missed the exam and called the professor with the excuse that they had been out of town, had a flat tire, and were stranded on a desolate highway. The professor agreed to give them a make up. At the appointed time, the professor gave each student an exam and put them in separate rooms. On page 1 was a question worth 5 points for which they were well prepared. On page 2, they found the following question worth 95 points: Which tire? What is the probability that the first student and the second student will guess the same tire? If you use Fraction Tiles to solve the problem, let the clear tile represent all the possibilities.

Problem 2: Reynaldo and Reya’s Excellent Adventure

Reynaldo and Reya’s father, the King of Lackerby, had fallen under the spell of the wily, magical salamander named Newt. To free their father and the whole kingdom from the power of Newt, the brother and sister sought the advice of the wise wizard who lived in the forest. “To break the spell on your father,” the wizard instructed, “you must—before the sun sets—find the magical ring of truth hidden deep in the forest beyond.”

1. The siblings had no sooner begun their search for the magical ring when the forest trail divided into three paths (see figure below). (a) Illustrate how an area model can be used to illustrate this situation. (b) What is the probability of taking the left path (Path 1 in the figure below)?

2. Unsure what path to choose, the siblings stood frozen in place as precious minutes ticked by. Then Reynaldo recalled the wizard’s first clue, “One of three will not your father free. It leads to Grid Loch, where the road is all but blocked.” After a moments thought, he concluded, “The clue means that we can’t take one of these three paths. It must also mean that one or another of the remaining paths should lead to the magical ring. But which one?” (a) How would an area model illustrate this situation? (b) What is the probability of the siblings choosing one correct path or the other?

3. "No Reynaldo, that’s not right," reminded Reya. "The wizard’s riddle said, ‘The middle of three will not your father free.’ The siblings decided to take the left path (Path 1). Deep in the forest, just before sunset, this path split into three branches. “There were two more clues,” recalled Reynaldo. “Yes,
Investigation 13.E continued

‘the right of two leads to goo’ and ‘The right of three is what you need,’” exclaimed Reya. "The right path [Path 3 in the figure] we didn’t take must have two branches, and its left branch [Branch 3A in the figure] must lead to the ring. And the right-most branch of the left path we took [Branch 1C in the figure] must also lead to the ring.” Illustrate how an area model could represent the following situation: Given that a traveler was already on Path 1, what is the probability of choosing Branch 1C?

4. If the sibling had not deciphered the wizard’s clues and simply guessed at each fork in the trail, (a) what is the probability that they would have chosen Path 1 and then Branch 1C that lead to the ring of truth? (b) What is the probability that the siblings would have chosen Path 3 and then Branch 3A that also lead to the ring? (c) What is the probability they would have chosen either Path 1 and then Branch 1C or Path 3 and then Branch 3A and arrived at the ring?

Problem 3: The Problem of Points

Shawn and Shana were playing a game. Behind his back, Shawn hid a penny in one hand. Presented two closed fists, Shana had to pick the hand with the penny. If she did, she scored a point. If she picked the empty hand, Shawn got a point. The first player to get 10 points won.

1. Illustrate how an area model could represent the probability of Shana choosing a hand at random and picking the hand with the penny.

2. Illustrate how an area model could represent the probability of Shana randomly picking a hand with a penny at least once in two tries.

3. One day the score was Shawn 8 and Shana 9 when the players had to stop because their school bus had arrived. The next day, the players decided to finish their game. (a) What is the probability of Shana winning? (b) Marc argued that the answer to the previous question was \(\frac{1}{2}\). Does his answer make sense? Why or why not? (c) Illustrate how an area model could represent the probability of Shawn winning the game. Hint: With the score 9 to 8, what are the possible outcomes of the next flip and the probability of each? (d) What is the probability of Shawn winning? (e) Does your answer to the previous question square with answer to Question 3a above? Why or why not?

Problem 4: A Spelling Contest

Brittany knew how to spell \(\frac{1}{2}\) of the words in her spelling book. Burgundy knew \(\frac{2}{3}\) of the words. The girls decided to have a spelling contest in which they would alternate spelling words until one won by spelling a word correctly. To be fair, Burgundy let Brittany go first and randomly chose one of their spelling words. Illustrate how an area model could be used to answer the following questions:

1. What is the probability of Brittany winning on her first try?

2. (a) What is the probability of Brittany misspelling her first word and Burgundy winning on her first try? (b) Does she stand a better, a worse, or an equal chance of winning than did Brittany?

3. What is the probability of Burgundy misspelling her first word and Brittany winning on her second try?

4. What is the probability of Brittany misspelling her second word and Burgundy winning on her second try?

5. (a) On the second try, who has the better chance of winning? (b) Compared to the first try, did the gap between the probabilities of each girl winning increase, decrease, or stay the same? (c) What would you predict would happen to the gap on the third try?

e. Francisco wanted to represent the number of children with 0 to 4, 5 to 9, 10 to 14, 15 to 19, and 20 to 24 books read in a semester. He should use a **bar graph** to represent these data.

f. A **line graph** is useful for representing relative amounts.

g. A **circle or pie graph** is useful for illustrating percents up to 100%.

3. In the game *Mastermind*, how many possible combinations could be used to create 4 marble combinations from 6 colors? A color may be used more than once.

4. From the starting position on the board below, you may move diagonally but not straight up, down, left or right. For each other position on the board determine the minimum number of moves required to get to the position. Then determine how many different routes with these minimal moves are possible. Record this number on the position. What patterns do you notice?

   ![Example](image)

5. In Canada and Great Britain, “zip codes” include letters. Is there any advantage to using letters in a zip code?

6. Try solving the following conditional probability problems:

   **Problem 13.A: The Marble Argument Revisited** (8). Monique and Unique were again arguing over who owned two special marbles. To end the rife, their father put the two special marbles in a bag along with two regular marbles. The girls flipped a coin to see who would blindly choose two of the four marbles. Monique won. (a) Monique pulled out a special marble on her first try. What is the probability of doing so on her second try? (b) What is the probability of picking a special marble on her first try given that her second pick was a special marble?

   **Problem 13.B: The Odds Be With You** (8). A store used the following promotion for its opening: The first 100 customers will each be given a chance to choose one of three cards from a box. One card will have a gift certificate on each side, another will have a gift certificate on only one side, and the third will be blank on both sides. If Arillio pulled out a card with a gift certificate on one side, what is the probability that the other side has a gift certificate also?

7. (a) What is the probability of flipping a coin and getting two tails in a row? (b) What is the probability of getting at least one head in those two flips? (c) How are the answers to the previous two questions related? (d) What is the probability of flipping a coin and getting three tails in a row? (e) What is the probability of getting at least one head in those three flips? (f) How are the answers to the previous two questions related? (g) What is the probability of flipping a coin and getting four tails in a row? (h) What is the probability of getting at least one head in those four flips? (i) How are the answers to g and h related?

8. Is the solution of Problem 3 of Investigation 13.8 on page 13-25 of the *Student Guide* related to sample size? Why might this problem mislead many people?

9. In *The I Hate Mathematics Book* by Marilyn Burns (©1975, Little, Brown and Company, Boston) **combinations** and **permutations** were defined using an ice cream cone analogy—a situation where a person can have a cone with 2 scoops of ice cream and can choose from 3 flavors of ice cream. For both terms, Burns’ definition allows for repetitions (e.g., two scoops of chocolate are counted as a combination and a permutation). Combinations are further described as the number of possibilities when order does not matter (e.g., chocolate on top and vanilla on bottom is treated the same as
vanilla on top and chocolate on bottom). Six combinations were illustrated. Permutations are referred to as the number of possibilities when order is considered (e.g., when chocolate on top and vanilla on the bottom is treated as different from vanilla on top and chocolate on the bottom). Nine permutations were illustrated. Are these definitions of combinations and permutations consistent with those described in Box 13.1 below? Why or why not?

**SUGGESTED ACTIVITIES**

1. (a) Collect newspaper, magazine, or internet articles that illustrate the everyday use of statistics and/or probability. (b) Find examples of the misuse of statistics or probability and explain how. Note that this would make an excellent project for intermediate-level students and could serve as the basis for a bulletin board.

2. (a) Describe your own idea for a bulletin board display involving statistics or probability. Indicate the learning aims of the display and illustrate how it might look. (b) Implement your bulletin-board plan. (c) Evaluate its effectiveness. That is, document interest in the display and what children learned from it. Summarize your results with appropriate graphs and statistics. (d) Present your idea, data, and conclusions to your class.

---

**Box 13.1: Counting Techniques**

As counting techniques, the Fundamental Counting Property, combinations, and permutations are useful in determining the number of possibilities and, thus, in determining probabilities.

- **The Fundamental Counting Property.** According to this property, the number of possible ways two events can occur in distinct ways (e.g., in succession) can be determined by multiplying the number of ways each separate event can occur. For example, the total number of outcomes (sample space) for flipping a coin twice is $2 \times 2$ or 4 (the number of ways the first flip of the coin can occur, 2, times the number of ways the second flip of the coin can occur, 2). Note that, for each flip (event), there are two distinct outcomes or possibilities (heads or tails) and that these events occur successively (the coin is flipped once and then again). The total number of outcomes for flipping a coin and rolling a six-sided die is $2 \times 6$ or 12 (the number of outcomes of the coin flip, 2, times the number of outcomes for the die roll, 6).

- **Advanced Counting Techniques.** Like the Fundamental Counting Property, the advanced counting techniques of combinations and permutations apply to mixing two or more sets of distinct objects (e.g., different colors, numbers, letters, or names). Unlike the Fundamental Counting Property for determining the total number of possible outcomes, combinations and permutations do not include repetitions. For example, if three children (Arlette, Brian, and Cal) are eligible for two positions (morning monitor and afternoon monitor), it follows from the Fundamental Counting Property that there are 3 x 3 or 9 possible outcomes including Arlette-A.M. and Arlette-P.M., Brian-A.M. and Brian-P.M., Cal-A.M. and Cal-P.M. Combinations and permutations would exclude these repetitions. That is, they would include only those possibilities where the monitors in the morning and afternoons were different people (e.g., Arlette-A.M. and Brian-P.M.).

  **Combinations involve counting the number of mixtures without regard to the order of the elements in the mixture.** Consider, for example, determining the probability of randomly choosing the letters N and O from the choices of M, N and O. In all, there are three combinations of these letters (MN, MO, and NO) and so the answer to our question is $\frac{1}{3}$. Note that once a letter was picked, it could not be picked again (e.g., MM is not a mixture). Moreover, note that the order of the letter was unimportant. It did not matter whether N was picked first and then O or whether O was picked first and then N.

  **Permutations involve counting the number of mixtures taking into account the order of the elements in the mixture.** Consider, for example, determining the probability of randomly picking letters from the choices of M, N, and O and spelling ON. In all, there are six permutations of these letters (MN, NM, MO, OM, NO, and ON) and so the answer to our question is $\frac{1}{6}$. Note again that nonmixtures such as MM were not counted. However, it clearly mattered whether an N or O was picked first and which was picked second.
3. M&M candies come in different colors. Are the colors equally represented or not? Collect a sample of the small bags of M&Ms. Chart your results. Note that this could serve as the basis for an investigation for either primary- or intermediate-level students.

4. (a) Propose a hypothesis about a physical event, a social situation, human behavior, or mathematical teaching or learning. Devise a study to explore the hypothesis. (b) Conduct the study. That is, collect and analyze data on the issue of concern. Draw conclusions from your results. (c) Write a report that describes your hypothesis, including a rationale for choosing the hypothesis (i.e., why it is worth exploring the issue); the study, including how the data were collected, what data were collected, and how they were analyzed; and your conclusions, including patterns or relationships suggested by your data, interpolations or extrapolations from your data, and/or significance or implications of the data. (d) Present your report to your class. Include appropriate tables and graphs, descriptive statistics, and/or inferential statistics in your report. Be prepared to defend your choices of data collection and analysis.

5. Volunteer to serve as a judge for a science fair in a local elementary, middle, or high school. Write a critique of at least two science projects. Evaluate the methods for collecting and analyzing the data. Were appropriate graphs used? Were appropriate descriptive or inferential statistics used? Justify appropriate uses. If applicable, describe how the data collection and analysis methods could have been improved.

6. (a) In terms of the instructional guidelines outlined in chapter 13 of the Student Guide, evaluate the statistics or probability unit or material in an elementary textbook for at least two grade levels of your choice. Identify whether the suggested instruction best fits the description of a skills, conceptual, investigative, or problem-solving approach. (b) Present your findings and conclusions to your class using appropriate graphs and statistics.

7. (a) Devise a lesson that involves one of the reader inquiries in chapter 13 of the Student Guide. (b) Try out and, if possible, videotape your lesson with a developmentally ready group or class of intermediate-level children. Assess your pupils’ learning, and evaluate the strengths and weaknesses of your plan and lesson. (c) Share with your class a summary of your lesson plan and the results of your teaching experience.

8. Using Box 13.1, Activity File 13.1 or Investigation 13.1 in the Student Guide (pages 13-3, 13-4, and 13-5 and 13-6, respectively) as a model, devise an integrated lesson that could be used to introduce a statistical or probability concept. Specify the aims of the lesson for other content areas as well as for mathematics. Indicate how the lesson will create a need for the concept. (b) Try out (and videotape) your lesson with a group or class of elementary-level children. Assess their progress in constructing an understanding of statistics or probability. Evaluate the strengths and weaknesses of your lesson. (c) Share your lesson plan and your assessment of it with your class.

9. (a) Tutor a group of fourth-, fifth- or sixth-graders for at least three one-hour sessions over the course of at least two months on statistics or probability. (b) Document your tutees’ progress. In particular, describe any informal strategies they invented to solve problems and their symbolic-level shortcuts. (c) Present your results and conclusions about them to your class.

10. (a) Using publishers’ catalogues, reference books on teaching, the internet, or other sources, make a list of instructional resources that would be helpful in teaching statistics or probability. (b) Obtain one or more of the resources. Evaluate a resource in terms of what approach to mathematics instruction it seems to suggest. Indicate how one or more recommended activities could serve, or be adapted to serve, as a worthwhile task and the basis for the investigative approach. (c) Share your data, evaluation, and teaching ideas with your class.

11. (a) Find examples of children’s literature in which statistics or probability play a key or interesting role. (b) Consider how a lesson or unit that embodies the investigative approach could be built around one or more of the books on your list.
12. (a) Try out one of the examples of educational software listed on page 13-32 of the Student Guide or an educational program of your own choosing. (b) Evaluate the software in terms of its ease of use and instructional value. (c) Share your evaluation with your class, including a demonstration of particularly interesting or useful aspects of the program review.

13. Visit the Math League web site on percent and probability (http://www.mathleague.com/help/percent/percent.htm) or a web site of your own choice. (a) Evaluate the web site as a resource for intermediate-level students. Consider, for example, the clarity with which key probability terms are and extent to which connections are made to everyday applications of probability or other mathematical content. (b) Evaluate the value of the web site for a student who is having difficulty understanding probability. (c) Share your analyses of the web site with your class.

16. (a) Choose a relatively simple everyday situation that intermediate-level students might find interesting to simulate and devise a simulation in order to make predictions about it. (b) Carry out your simulation a number of times. Collect, analyze, and describe your data. Indicate what predictions your simulations suggest. (c) Write a program for your simulation (see, e.g., Box 17.7 on page 17-20 of the Student Guide).

HOMEWORK OR ASSESSMENT

QUESTIONS TO CHECK UNDERSTANDING

1. The following set of 31 scores has a bimodal distribution because it has two peaks: 0, 0, 1, 1, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 7, 8, 8, 8, 9, 9, 9, 10, 10, 10, 11, 11, 12. (a) Would a mean be an accurate average (measure of central tendency)? Why or why not? (b) Would a median be an accurate measure of central tendency? Why or why not? (c) What would be the most accurate way to describe the central tendency of these data?

2. Shown below is the silhouette of a roller coaster. On a piece of graph paper, graph (a) the height of a roller coaster car over time and (b) its speed over time. (c) Would children find Questions 2a and 2b equally difficult? Why or why not?

3. Graeme got the lowest score on his class’ unit test. Dian got the highest score. If Graeme scored 37 points, the mean for the class was 70 points, and the range was 63 points, what was Dian’s score? (Based on a NAEP problem.)
4. (a) Find the missing number for \( x \) so that the mean of 17, 9, 11, and \( x \) is 16. (b) Describe three different strategies for answering this question.

5. Sonny had quiz grades of 9, 8, 7, 7, 6, 6, 6, 5, 5, and 0. He complained to Miss Brill that his grade of F was unfair. Miss Brill pointed out that he had a failing average. Sonny countered that he had a median score of 6, which was passing. Does Sonny have a leg to stand on here? Briefly justify your answer.

6. After learning about a leveling analogy (a mean can be thought of as the height of scores after all had been evened out), a baseball fan asks you, "How does that apply to a baseball average such as .250?" (A baseball average is the ratio of hits to at bats.) Could a batting average be considered a mean? Why or why not?

7. A student asks, "Why do you add all the numbers and divide the total by the number of numbers to determine the mean?" (a) According to the Student Guide, children can be helped to rediscover or understand the algorithm for computing a mean by relating it to what meaningful analogy? (b) Illustrate how a teacher could use this analogy to help the student discover the answer to her question. Include an appropriate problem and illustrate how it could be solved concretely with blocks.

8. A newspaper reported that salary hikes for the faculty of a local university were expected to average 3%. It also reported that some faculty members recommended that the university administration report "what most faculty members would get, rather than an average percent increase, so that no one would get false hopes." What measure of central tendency were the faculty suggesting the administration use? Justify your answer.

9. The owners of baseball teams argued that the average salary for a major league player was $3,000,000. The players' union representatives countered that the average salary was, in fact, only $2,000,000. Both sides checked their statistics, found they were correct, and charged the other with providing false information. (a) Statistically speaking, how might this misunderstanding have arose? (b) As a mediator, how could you help resolve this conflict?

10. After a month of formal instruction, intermediate-level students were asked to solve the following problem:

   ■ A Long Shot (4-8). Sabine and her two friends decided to play a game. They agreed that whoever rolled the largest number on a die numbered from 1 to 6 would go first. Sabine was the first to go and rolled a one. What is the probability that both of her friends will also roll a one and thus not beat her?

   (a) Common solutions were \( \frac{2}{6} \) and \( \frac{2}{12} \). Why might students make each of these errors? (b) How could a teacher help students recognize that the solutions of \( \frac{2}{6} \) and \( \frac{2}{12} \) do not make sense? (c) What is the correct solution? Justify your procedure and solution.

11. (a) What is the probability of drawing a diamond from a standard (52-card) poker deck? (b) What is the probability of drawing a diamond or a heart? (c) What is the probability of drawing two diamonds in a row with replacement (if a card drawn previously is replaced)? (d) What is the probability of drawing two diamonds in a row without replacement (if a card drawn previously is not replaced)? (e) What is the probability of drawing three diamonds in a row with replacement? Express probability in terms of fractions. Reduce fractions when possible.

12. Willy B. Rite has been selected to participate as a contestant on a television game show. If he is lucky enough to win, he will be able to bring home a brand new BMX bicycle. If he loses, he goes home empty-handed. The game show involves playing Draw Three to Win. Five checkers (two black, three red) are concealed in a bag. The object of the game is to draw out the three red checkers, without ever selecting a black checker. Once a checker is drawn, it is set aside. If the black checker is drawn, the player automatically loses. (a) What is the probability that Willy will pull a red checker out of the bag on the first draw? (b) What is the probability that Willy will select a red checker on each of his three picks and, thus, win a new BMX bike? (c) What type of proba-
bility situation does the previous question represent? (d) Illustrate how Question b could be solved informally. (e) Unfortunately, Willy pulled out a black checker and did not win the new BMX bike. But the MC gave Willy another chance. In Draw Two, there is just one black checker and one red checker in the bag. Willy's job is to pull out a red checker two times in a row. Obviously, he has to replace whatever he draws from the bag on a turn before he takes the next turn. Illustrate two ways students could concretely represent and informally determine the probability of Willy winning.

13. Circle the letter of any statement below that, according to the text, is true. Change the underlined portion of any false statement to make it true.

a. Initial lessons on collecting data should not draw distinctions between categorical and numerical data.

b. Children commonly have difficulty with graphs when the graphic representation is different from their visual image of the situation.

c. Graphing instruction should focus on how to construct graphs.

d. Statistics instruction in the elementary grades should primarily involve analyzing realistic sets of data.

e. Descriptive statistics involve using probability to gauge whether or not it makes sense to apply a conclusion about a sample to a general population.

f. Graphs provide a visual display which makes relationships more transparent. This is important when solving problems or communicating data but not when making predictions.

g. A mean is an accurate and appropriate measure for the central tendency of the following data (snowfall over a 2-week period): 0", 0", 0", 0", 0", 0", 0", 0", 0", 0.1", 0.1", 0.2", and 18".

h. Circle graphs are a useful way of intuitively introducing the concept of correlation.

i. A large sample ensures that it is representative of a population.

j. Children just beginning school have a good intuitive probability sense.

k. Children in Level 3 can (1) recognize that the probability of picking a blue marble is the same if there are two blue and two red marbles or four blue and four red marbles and (2) informally determine all the possible outcomes of flipping two pennies.

l. A common difficulty children have in solving probability problems such as determining if a game is fair (see Investigation 13.6 on page 13-20 of the Student Guide) is not recognizing that different outcomes can be equally likely.

m. Groundwork instruction on probability with primary-level children should not include solving problems involving combinations and permutations.

n. Groundwork instruction on probability with primary-level children should avoid the use of probabilistic terms such as certain, likely, and unlikely.

o. Elementary-level probability instruction should begin with determining theoretical probabilities informally, proceed to determining theoretical probabilities formally, and end with gauging empirical probabilities.

14. For each of the following problems, (a) describe a simulation, (b) determine the solution using your simulation, and (c) determine the solution using theoretical probability.

- **Problem A: Seriously into Cereal (5-8)**
  If one Beasts-of-Horror-Sure-to-Make-Any-Adult-Sick ring is randomly placed in each box of Cereal-Coated Sugar Crystals cereal and there are six different rings, how many boxes could Orgo realistically expect to buy to get all six rings?

- **Problem B: Rainy Cities (5-8)**
  If there is a 50% chance of rain in 12 different cities in different parts of the world, how many cities would you expect to have rain?
Inada had been looking forward to her cousin’s wedding day. But severe storms had downed trees and caused flooding. A TV report indicated that half the roads in the area were impassable. Using the map below, gauge Inada’s chances of finding an open route to her cousin’s home. (Assume no back tracking.)

![Map of towns](image)

Problem D: Traveling Toddler (♦ 5-8).
After his babysitter left, two-year old Avis noticed that the screen door had not yet completely shut. The toddler pushed open the door and ran for freedom. Fifteen minutes later his shocked parents discovered he was not home. If at each corner, Avis haphazardly chose a direction (north, east, south, or west) to go and it took him 5 minutes to walk from corner to corner, how likely is it that Avis was (i) no more than 1 corner away, (ii) exactly two corners away, (iii) exactly three corners away, and (iv) more than three corners away?

Problem E: Test Savvy (♦ 5-8). The students at the Ivy Hill School had to take an achievement test. Part I of the test consisted of 20 multiple choice questions with 5 possible choices each. If a correct answer is scored 5 points, no answer is scored 0 points, and an incorrect answer is scored -2 points, in which of the following situations is there an advantage to randomly guessing a choice: a student can eliminate (i) three choices as incorrect, (ii) two choices as incorrect, (iii) one choice as incorrect, and (iv) none of the choices as incorrect?

15. (a) Consider determining the probability of a repetitive event such as obtaining a head on successive coin flips or obtaining a six on successive rolls of a six-sided die numbered 1 to 6. Devise a formula for determining the probability of obtaining the same outcome for any given number of repetitions of an event. (b) Consider "at least" problems such as that in Box 13.1, Problem 4 of Investigation 13.7, and those in Investigation 13.8 (pages 13-3, 13-21, and 13-25, respectively, in the Student Guide). Devise a general formula for determining the probability of a particular outcome for such situations.

WRITING OR JOURNAL ASSIGNMENTS

1. You want to introduce your first-grade class to graphing. Your student teacher Priscilla proposes the following lesson:

   Today, class we’re going to learn all about making bar graphs. Look at the handout (shown below) I put on your desk. At the top, it shows blue (B), red (R), and green (G) balloons. Count the number of blue balloons. Now label the top line Blue. Color in the same number of boxes that you just counted

   ![Handout](image)

---

Based on the Five Pumps problem from the Quantitative Literacy Project.
Evaluate Priscilla’s graphing lesson in terms of the instructional guidelines outlines in this chapter. Include in your evaluation what her lesson focused on and what type of graph she proposed to introduce graphing. What suggestions would you make for improving her lesson?

2. For each of the following situations, describe the reason for a child’s response and how a teacher could help students construct an understanding of the concept or procedure involved.

a. The class had measured the amount of snow fall each day for a week—Monday, 6”; Tuesday, 4”; Wednesday, 4”; Thursday, 1”; and Friday, 0”. Isador did not understand why the sum of 15” had to be divided by 5 rather than 4. He also did not understand how the answer could be 3 rather than one of the scores.

b. Minerva did not understand why summing the differences between each score and the mean had to be zero.

c. Assume that Isador and Minerva did not understand the first model (analogy) you used. Explain how exploration with a different model might help.

d. Keeley argued that her group’s sample for a school opinion poll on what books to buy should consist of half boys and half girls so as to be fair to both. The school had 460 boys and 340 girls.

e. For two 0 to 5 dice, a class concluded that the sample space consisted of 11 outcomes (the sums 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10) and that the probability of each was \( \frac{1}{11} \).

f. To determine the probability of flipping a fair coin and getting two heads, Rudy answered \( \frac{2}{2} \).

3. In the introduction to chapter 13 of the Student Guide, Miss Brill can’t believe that new instructional guidelines recommend that statistics and probability be introduced in the elementary grades, in part, because she considers these topics too difficult for children. How could you justify teaching statistics and probability at the elementary level to skeptical colleagues, administrators, or parents. Include in your justification why these topics are important and why they are important to teach at the elementary level. Also, address Miss Brill’s concern about their difficulty.

4. Describe how statistics or probability affects your everyday life.

5. You are serving as a judge for a middle school science fair. Jack explored whether lima bean plants grew better under sunlight or equal amounts of lamplight. He included Figure 13.1 below in his exhibit. Jack reported that the mean height of the 12 bean plants in the sunlight condition was 2.13 and that for 12 bean plants in the lamplight condition was 2.2. He concluded that lima bean plants grew better under lamplight. Evaluate Jack’s conclusion.

PROBLEMS

■ Mean of a Lot of Numbers (◆ 6-8)

What is the mean value of the numbers 1 to 2000?

■ A Dearth of Stopwatches (◆ 6-8)

As an official for Athletic Day you have been asked to determine the median time of the runners in a 2-mile race. There are 120 runners in the race. What is the minimum number of stopwatches you need to check out from the gym office to determine the median? Assume that a watch will be used to time one and only one runner.

■ A Passing Grade (◆ 6-8)

Henri had scored 20, 12, and 16 correct of a

<table>
<thead>
<tr>
<th>Figure 13.1: Results of Jack’s Science Fair Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant Heights for the 12 Plants in Each of Two Lighting Conditions</td>
</tr>
<tr>
<td>sunlight:</td>
</tr>
<tr>
<td>lamplight:</td>
</tr>
</tbody>
</table>
possible 24 on each of his first three spelling quizzes of the semester. If he got a passing grade (had an average of 75% or greater), his father promised him $5. What is the minimum average number correct Henri must achieve to get a passing grade and get $5? What must Henri score on the fourth and final spelling quiz of the semester to achieve this minimum average? Does Henri have a chance of passing?

A Re-averaging Problem (6-8)

Ms. Brill used a calculator to determine the mean of her students’ 12 quarterly quiz grades (see Figure 13.2 below). After discussing the twelfth and last quarterly quiz with her class, Miss Brill realized that several questions were ambiguous and that the quiz should be thrown out. This meant she had to recompute the quiz averages for the quarter. There must be an easier way to determine the new averages, thought Miss Brill. Is there a shortcut for determining the average of Quizzes 1 to 11 for each student? Hint: Consider how she could use the average for quizzes 1 to 12 to determine the new average.

A Re-averaging Problem (6-8)

A Re-averaging Problem (6-8)

A Re-averaging Problem (6-8)

Fives (4-8)

Given the natural numbers 1 to 100, what is the probability of randomly selecting a number with at least one 5?

A Chancy Checking Method (4-8)

Arc flipped a coin to determine his answers to a true-false test—marking true if the coin came up heads; marking false if it came up tails. After the rest of the class had finished, Arc was still at work. Asked what was taking so long, Arc responded that he was just checking his answers—as the teacher had stressed. If he checked his answers by flipping a coin, what is the probability that on any given item, Arc will change his answer?

A Dog’s Luck (6-8)

Ruffus the dog was home alone. Smelling smoke, he went to the telephone and—as he had seen on T.V.—tried to dial 911. (a) What is the probability of Ruffus hitting a 9, a 1, and a 1 (not necessarily in that order) if he paws the phone three times and each time randomly presses a phone button? The phone Ruffus used has 12 buttons (0 to 9 buttons plus * and # buttons). (b) What is the probability of correctly dialing 911—of dialing 9, 1, 1 in that order?

Unlucky Sibling (6-8)

Kirk has the same number of brothers as he has sisters. His sister Andrea has twice as many brothers as she has sisters. One of the siblings is randomly selected to empty the garbage. What is the probability that a boy will be chosen? What is the probability that Kirk will be chosen?

A Really Foul Forecast (5-8)

Myrna was looking forward to a pleasant three-day weekend. She planned to drive to her friend’s lake cottage Saturday and return home Monday. Unfortunately, there was 50% chance of rain Saturday, 66\( \frac{2}{3}\)% chance of rain Sunday, and a 75% chance of rain Monday. (a) What is the probability Myrna will drive to and from the cottage in a rain storm? (b) What is the probability of Myrna driving in a rain storm on the way to or from the cottage (but not both ways)? (c) What is the probability of it storming on at least one of the days she is driving? (d) What is the probability she will have clear driving both days? (e) What is the

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hector</td>
<td>88</td>
<td>90</td>
<td>100</td>
<td>98</td>
<td>80</td>
<td>86</td>
<td>85</td>
<td>90</td>
<td>90</td>
<td>76</td>
<td>92</td>
<td>56</td>
<td>85.9</td>
</tr>
<tr>
<td>Jose</td>
<td>90</td>
<td>92</td>
<td>96</td>
<td>96</td>
<td>95</td>
<td>90</td>
<td>94</td>
<td>100</td>
<td>88</td>
<td>94</td>
<td>96</td>
<td>62</td>
<td>91.1</td>
</tr>
<tr>
<td>LeMar</td>
<td>90</td>
<td>96</td>
<td>100</td>
<td>95</td>
<td>100</td>
<td>98</td>
<td>100</td>
<td>100</td>
<td>90</td>
<td>92</td>
<td>100</td>
<td>60</td>
<td>93.4</td>
</tr>
<tr>
<td>Sandra</td>
<td>92</td>
<td>94</td>
<td>96</td>
<td>90</td>
<td>100</td>
<td>92</td>
<td>94</td>
<td>93</td>
<td>96</td>
<td>98</td>
<td>90</td>
<td>55</td>
<td>90.8</td>
</tr>
<tr>
<td>Violet</td>
<td>85</td>
<td>82</td>
<td>80</td>
<td>80</td>
<td>90</td>
<td>88</td>
<td>84</td>
<td>82</td>
<td>84</td>
<td>90</td>
<td>80</td>
<td>48</td>
<td>81.1</td>
</tr>
<tr>
<td>Yuri</td>
<td>80</td>
<td>88</td>
<td>90</td>
<td>75</td>
<td>72</td>
<td>82</td>
<td>84</td>
<td>88</td>
<td>70</td>
<td>80</td>
<td>82</td>
<td>56</td>
<td>78.9</td>
</tr>
</tbody>
</table>
probability of it storming all three days of the weekend? (f) What is the probability of it storming at least one of the three days of the weekend?

A Goofy Game? (♦ 5-8)

A student commented that Dice Baseball (Activity File 1.2 on page 1-20 of the Student Guide) was not very realistic because batters did not get hits often enough. (a) Is this game unrealistic as charged? Why or why not? Hint: What is the probability of getting a hit (a single, double, triple, or home run) in Dice Baseball? Recall that this game involved rolling two dice and summing the two numbers and that a roll of 8 = a single, a roll of 10 = a double, a roll of 2 = a triple, a roll of 12 = a home run, a roll of 4 = walk, and any other sum = an out. Some things you need to know about baseball are:

• A decent batting average in major league baseball is about .275 (275 hits in every 1000 at-bats).

• Walks do not count as a hit or as an at-bat.

(b) One group of students concluded that the probability of getting a hit was \( \frac{4}{10} \) or .400 and that the game was unrealistic because batters got hits too often. How did the group arrive at this probability? Was this group correct or not? Why or why not? (c) Revise the rules for the Dice Baseball game for play with two 10-sided die (0-9). List what sum would represent each of the following outcomes: a single with base runners advancing two bases, a single with base runners advancing one base, a double with base runners advancing three bases, a double with base runners advancing two bases, a triple, a homerun, a walk, a sacrifice fly (batter is out but runners advance one base), an out, a double play (defense picks which offensive players are out), a triple play (defense picks which offensive players are out). Constraint: The probability of getting a hit should be about .275.

Probability Maze (♦ 5-8)

The NCTM (1989) Curriculum Standards includes the following problem: By randomly choosing paths, what is the probability a traveler will end up in room A? (Assume that once a traveler ends up in Room A or B, no further travel is possible.) Jonas’ group concluded it was 2/5. Does this answer make sense? Why or why not?

Roundabout (♦ 5-8)

Roundabout is an interesting dice game for two players that can be used to practice basic addition facts in the primary grades. It could also be the basis for problems in probability for older students. A gameboard like that shown below can be readily made. The only other materials needed are dice with 1 to 6 dots (or numerals 1 to 6) and two tokens. To advance to a position, one or both dice must equal the number at the position. To get to Position 1, a player would have to roll a 1 on a die. To get to Position 2, a player would have to roll a 2 on a die or a sum of 2 on both dice. To get to Position 3, a player would have to roll a 3 on one die or a sum of 3 on both dice. The first player to get back to the START is the winner. To move from Position 11 to the START position, a player has to roll a 12. If a player cannot advance to the next position, the player’s turn ends and the opponent rolls. If a player lands on a space occupied by an opponent, the opponent must return his or her token to the START position and begin again. Play the game once to get a feel for it. Then answer the following questions. Illustrate how children could answer each question informally.

Situation I
1. If Adie is at the START position. She throws out a die. What is the probability she will roll a 1 and be able to advance to Position 1?

2. What is the probability she will roll a 1 on either of the two dice thrown?

Situation II
1. Adie is on Position 2, what is the probability
she will roll 1, 2 or 2, 1 (dice summing to 3) and be able to advance to Position 3 on her next roll?
2. What are her chances of a roll that shows a 3?
3. What are her chances of rolling 1, 2; 2, 1 or a 3?

Situation III
1. Adie is on Position 11 and about to take her turn. What is the probability she will win (roll a 12) on her next roll?
2. What is the probability Adie will roll a 12 on either of her next two rolls?
3. Given the rules of the game, if Adie is on Position 11 and Bette is on Position 10, theoretically who is in a better position to win. Justify your answer.

Situation IV
1. If Adie was on Position 11 and Bette was on Position 8 and Adie failed to roll a 12 in her next three rolls, Bette could conceivably overtake her by rolling a 9, 10, and 11 in that order. Is this problem of finding the probability of independent events occurring together or a problem of conditional probability? Briefly justify.
2. Estimate Bette’s chances of overtaking Adie. Who is more likely to win?

Looking for Patterns
1. A player stands the greatest chance of moving from which position to which position? Briefly justify.
2. A player stands the least chance of moving from which position to which position? Briefly justify.

Flip Soccer (♦ 5-8)

While in detention, Shana amused herself by playing Flip Soccer against herself. She began by placing a marker (the soccer ball) in the center of the pencil-drawn field. Shana then flipped a coin. If the coin came up a heads, she moved the marker one line to the right toward her goal. If it came up tails, she moved the marker one line to the left toward the dreadful Mr. Bailey’s goal. (Mr. Bailey had assigned her detention). (a) What is the probability of Shana scoring in the first four flips? (b) What is the probability of her being one space away from her goal after four flips. (c) After four turns, what is the probability of the marker being at the center of the field?

Mr. B’s Goal O | | | | | | Shana’s Goal

The Odds Be Against You (♦ 7-8)

What are the chances of winning the Illinois State Lottery? In this lottery, a player must correctly pick 6 of 54 possible numbers (1 to 54). (The order of the numbers is unimportant.)

Lucky Racers? (♦ 7-8)

Ms. Caldwell-Green’s fifth-period class had an ongoing debate about whether, overall, boys or girls were better athletes. Tired of ceaseless unfounded speculation, the teacher proposed that the class collect some hard evidence on the issue. The class agreed to test the issue directly by having a track meet. While organizing the races, Keana suggested that the teams pick their two fastest runners. Swoozie noted that this would not be the best way to test whether, overall, boys or girls were the fastest runners. She proposed that each team randomly pick two runners for each event. Her suggestion was adopted. In the 50-yard dash, girls finished first and second and boys finished third and fourth. Mikal argued that the results proved nothing because the girls just got lucky picks. What is the probability of two girls taking first and second place by chance alone?

Contestant’s Choice (♦ 7-8)

You are a contestant on a TV game show. There are three identical appearing boxes. One box contains a bar of gold worth thousands of dollars. The other two boxes each contain a single penny. You are allowed to select one box and keep its contents. You select a box but before opening it, the host of the show makes a new offer. He opens one of the two remaining boxes and shows you that it contains a penny. You may keep the box you first chose or exchange it for the remaining unopened box. What should you do?

Jealous Twins (♦ 8+)

Nortiki took her sister’s twins on an outing to the neighborhood mall. The twins spotted a gum-ball machine and pleaded for a gumball. Nortiki knew that the twins were so insanely jealous of each other that if one got a particular color gumball, the other would want that color also. If there were 12 red, 6 white and 4 blue gumballs left in the machine, what is the probability the twins would get the same color?

† Based on a problem called Monty’s Dilemma described by Shaughnessy (1992).
ANSWER KEY for Student Guide

Key for Investigation 13.1 (pages 13-5 and 13-6)

The relationship between height of an incline and the distance a marble will roll is curvilinear. For an explanation, see page 353 of this guide.

Questions for Reflection

1. (a) No. (b) You could check roll distances for somewhat taller and shorter heights.

2. (a, b, & c) Yes. Both are numerical variables (data) and represent continuous quantities, which means they can have values "in-between" the whole numbers noted on the graph. (d) No. TV shows are a categorical variable and the number of people is a numerical variable representing a discrete quantity. In both cases, "in-between" or fractional values are not possible. (e) No.

Key for Investigation 13.2 (page 13-8)

Activity I

Upper-left graph → d (As it defrosts, the juice’s temperature slowly rises; its temperature suddenly rises with the addition of tap water; and then, left out, it gradually warms to room temperature.)

Upper-middle graph → b (As the baseball rises, it loses vertical speed because gravity slowly overcomes its upward momentum. At some point these two forces are exactly balanced and, for an instant, the ball loses all vertical speed. Then the ball begins to fall back to earth, gaining downward speed because of the acceleration due to gravity.)

Upper-right graph → e (Immediately heating the frozen microwave dinner results in a rapid increase in its temperature. Afterward it cools slowly until it reaches room temperature and its temperature stabilizes.)

Lower-left graph → a (The thrown ball rises in height until its upward momentum is overcome by gravity and it begins its descent toward the ground.)

Lower-middle graph → f (The retail value of the diamond drops appreciably as soon as the customer walks out of the jewelry store. Then, over a long period of time, the diamond slowly gains value. The valleys and bumps indicate a fluctuating market value.)

Lower-right graph → c (The tub slowly fills with water. It rises suddenly when the bather gets in the tub, drops suddenly when the bather gets out, and so forth.)

Key for Investigation 13.3 (page 13-11)

2. The height of people is strongly correlated to their arm span.

Key for Investigation 13.4: Activity I (pages 13-14)

2. \[ \frac{10 \text{marked (initial sample)}}{x} = \frac{2 \text{marked (second sample)}}{60} \]

\[ \text{total population} \]

\[ 2x = 10 \cdot 60 = 600 \]

\[ x = 300 \]

Key for Investigation 13.5: Activity I (pages 13-16 to 13-18)

The code and decoded message are:

a - h j o ( v Ø
b ? i * p ) w @
c > j q ! x ,
d = k : r $ y ^
e # l ; s + z <
f { m \ t %
g [ n / u &

5-10-95

If presented within the next two weeks, the bearer of this note has the right to spray water on their teacher on Field Day, ask their teacher to sing a song of their choosing (within the limits of social acceptability), require their teacher to walk backwards for half an hour, or to negotiate with me for any other privilege.

Educationally,
Your Principal
Key for Investigation 13.6 (page 13-20)

Game 1: Two Flips

2. Play the game a number of times. That is, determine the empirical probability of winning for each player.

4. Many elementary-level students, particularly primary-level children, incorrectly assume that each outcome (heads-heads, heads-tails, and tails-tails) is equally likely.

5. Useful problem-solving heuristics include making a list (Figure A below), constructing a table (Figure B below), or drawing a picture (namely a tree diagram such as Figure C below). In each case, children can readily see that the two-heads and no-heads players each has a one in four chance of winning and the one-heads player has a two in four chance.

<table>
<thead>
<tr>
<th>Flip 1</th>
<th>Flip 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Figure A: List

<table>
<thead>
<tr>
<th>Flip 1</th>
<th>Flip 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H-H</td>
</tr>
<tr>
<td>T</td>
<td>H-T</td>
</tr>
<tr>
<td>T</td>
<td>T-H</td>
</tr>
</tbody>
</table>

Figure B: Table

<table>
<thead>
<tr>
<th>Outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-H</td>
</tr>
<tr>
<td>H-T</td>
</tr>
<tr>
<td>T-H</td>
</tr>
<tr>
<td>T-T</td>
</tr>
</tbody>
</table>

Figure C: Tree Diagram

Key for Investigation 13.7 (page 13-21)

Problem 1: A Love Story

In the chart at the top of the next column, note that one person could have any 1 of 12 signs, as could the second person. Thus, there are 12 x 12 or 144 possible combinations in all. Of these 144 combinations, 12 combinations share the same sign. Therefore, the probability of two people meeting at random having the same sign is 12/144 = 1/12. It is, then, not a remarkable coincidence that Dick and Jane had the same sign.

Problem 2: The Marble Argument

(a) The probability of winning the coin flip is 1 in 2; of picking a special marble, 2 in 4; of both things happening together, \( \frac{1}{2} \times \frac{2}{4} = \frac{2}{8} = \frac{1}{4} \). (b) \( \frac{2}{4} \times \frac{1}{3} = \frac{2}{12} = \frac{1}{6} \).

Problem 3: Cube Confusion

(a) From the table below, it is clear that there are 16 possibilities, of which four involved a red and a yellow; probability = \( \frac{4}{16} = \frac{1}{4} \). (b) \( \frac{2}{16} = \frac{1}{8} \). (c) \( \frac{12}{16} = \frac{3}{4} \). (d) \( \frac{9}{16} \). (e) \( \frac{5}{16} \).

Problem 4: Foul Forecast

(a) From tree diagram A on page 389, it is clear that there is a 3 in 4 chance of it raining at least one day of the weekend. (b) As tree diagram B shows, there is a 5 in 9 chance.
Figure 13.3: A Portion of a Drawing for Solving "A Love Story"

Questions for Reflection

3. In Problem 2, for instance, children could reason that both event A and event B happening would be less likely than either happening alone. Thus, the probability of both occurring together should be less, not more, than the probability of each.

4. (a) No, and does not always mean multiply. This is yet another illustration that the key word approach to solving problems is misguided. (b) No. (c) Students need to think about the meaning of probability problems. (d) Solving probability problems requires careful and thoughtful reading of problems. Many people are mislead by Problem 1, for example, because they assume the problem involves finding the probability of randomly picking two Leos from a crowd of people. (Note that this is a different question from the one that was actually posed. The actual question asked what is the probability of having some sign in common, not a particular sign.) The probability of randomly picking two Leos can be calculated by multiplying the probabilities of each independent event (\( \frac{1}{12} \times \frac{1}{12} = \frac{1}{144} \)). This underscores how mathematics forces us to use language precisely and carefully.

5. Making a list, constructing a chart, or drawing a tree diagram or other picture (see, e.g., Figure 13.3 at the top of the next column).

Key for Investigation 13.8 (page 13-25)

Problem 1. Roll only once each turn. This will minimize the chance of rolling a 0 (\( p = \frac{1}{6} \)). Drawing a tree diagram would show that rolling the dice twice greatly increases the chances of getting a 0 at least once (\( p = \frac{11}{36} \approx \frac{1}{3} \)).

Problem 2. A tree diagram (see Figure 13.4 on the next page) shows that with two rolls of a 10-sided die, there are 51 possibilities where a number greater than 6 would come up at least once and 49 where it won’t. Thus, the 6 should be placed in a middle box.