TEACHING TIPS

AIMS AND SUGGESTIONS

Unit 1•1: Different Views

The aim of this unit is to help students reflect on their beliefs about the nature of mathematics, mathematics learning, and mathematics teaching and to construct a more accurate understanding about them. As a framework for analyzing their own beliefs, three views are compared and contrasted. Conventional beliefs about school mathematics are consistent with the back-to-basics movement and are often embodied in an authoritarian, decontextualized, and meaningless approach to instruction (the skills approach). The second view is consistent with the teacher-directed (teacher-centered) meaningful approach espoused by William Brownell (1935) (the conceptual approach). The third view is consistent with the student-centered and open approach championed by Piaget and radical constructivists (the problem-solving approach). A useful activity is to encourage students to analyze their elementary instruction in terms of these three approaches and consider whether or not they were satisfied with it and why.

Unit 1•2: New Directions

In this unit, the case for reforming mathematics instruction is made, general guidelines for fostering mathematics power are laid out, and the investigative approach is introduced. In our experience, prospective teachers have difficulty distinguishing between the conceptual and the investigative approaches and between the investigative and the problem-solving approaches. This is due, in part, because the investigative approach is a combination of the meaningful conceptual approach and the inquiry-based problem-solving approach. There are, however, important differences between the investigative approach and each of these other two suggestions for reforming mathematics instruction.

In particular, a teacher using the conceptual approach tries to impose understanding of symbolic (formal) procedures by explaining them with a manipulative model and having students imitate the manipulative model. In the investigative approach, a teacher would guide students’ construction of understanding of formal procedures by encouraging them to use what they know to devise their own manipulative models and, later, to devise symbolic procedures that parallel their manipulative model. In effect, a teacher encourages children to reinvent formal procedures or to devise their own alternative symbolic procedures that are equally or more effective than those traditionally taught in school.

Although more democratic and more supportive of student autonomy than the conceptual approach, the investigative approach is somewhat less democratic and more teacher-guided than the completely democratic and laissez faire problem-solving approach. Although teachers using this approach would invite students to present their own problems and would remain open to pursue interesting lines of inquiry that spontaneously arise, they have an overall plan that would include achieving mastery of skills and understanding certain key concepts (content aims), as well as fostering mathematical thinking and other competencies necessary for mathematical inquiry, such as problem solving, reasoning, communicating, and conjecture-making and -testing (process aims). Indeed, within the investigative approach the art of teaching entails finding or devising worthwhile tasks (e.g., projects or problems) that help students learn and practice needed content, while involving them in the processes of mathematical inquiry.

We feel it is important to distinguish among the skills, conceptual, investigative, and problem-
solving approaches for two reasons. One is that it provides a framework for critically analyzing curricula, textbooks, and teaching suggestions found in journals, presented in workshops, discovered on the internet, and so forth. For example, instructional materials, such as activity books or educational software, may embody the conceptual approach, or even the skills approach, instead of the investigative approach as implied by the advertising claims of a publisher.

A second reason is that it provides teachers a framework for reflecting on and guiding their own teaching and professional development. For example, many elementary teachers use manipulatives and small-group work, but in a manner consistent with the conceptual approach, not the investigative approach. These teachers (and the prospective teachers observing them) should be able to recognize that this form of instruction is better than the skills approach but does not meet all the ideals of the NCTM (1989, 1991) Standards. For example, it does not adequately promote mathematical inquiry. Moreover, many prospective teachers may first feel more comfortable making the relatively small step from the skills approach to the conceptual approach, rather than the relatively large step to the investigative approach. Nevertheless, these students should keep in mind that the conceptual approach does not fully embody the NCTM (1989, 1991) Standards and that their ultimate goal is moving on to the more complicated and demanding teaching required by the investigative approach.

Perhaps because of time limits, some instructors may wish to compare the investigative approach with only the skills approach. Comparisons with the conceptual and problem-solving approaches could then be done later in subsequent undergraduate or graduate courses.

To help students understand the rationale for the reforms proposed by the NCTM, encourage them to compare and contrast the skills and investigative approach in terms of the elements of mathematical power. Which is more likely to foster interest in mathematics, confidence, and self-reliance or autonomy (i.e., a positive disposition)? Which is more likely to foster mathematical thinking and competence in the processes of mathematical inquiry? Which is more likely to promote the understanding necessary for adaptive expertise and, hence, maximize transfer of learning to new tasks or problems and minimize forgetting and the need for constant review and repetitious practice?

### SAMPLE LESSON PLANS

Because constructing a general framework is so important, you may wish to spend more than one class period on chapter 1 material. For this reason, two-lesson lesson plans are illustrated below for the single-activity and multiple-activity approach.

### Project-Based Approach

Using the SUGGESTED ACTIVITIES on pages 26 and 27 of this guide as a menu, an instructor could assign one or more required projects, give students a choice of one or more required projects, and/or give them a choice of one or more extra-credit optional projects. Suggested Activities 1 and 3 could help students better understand and appreciate the goals set out in the NCTM (1989) Curriculum Standards. Suggested Activities 2, 5, 6, 9, and 10 could help them better understand and appreciate the investigative approach. Suggested Activities 3, 4, 6, 7, and 11 could be used to familiarize students with computer technology and resources available on the world wide web.

### Single-Activity Approach

#### Lesson 1. Use Probe 1.1: Examining Beliefs About Mathematics, Learning, and Teaching (pages 1-6 to 1-8 in the Student Guide) or Probe 1.4: The Investigative Approach (pages 1-28 to 1-32 in the Student Guide) to prompt students to reflect on and discuss (a) their beliefs about mathematics and its learning and teaching, (b) the need for reforming mathematics education, and (c) how the investigative approach differs from the skills approach (or other approaches) in philosophy (underlying assumptions) and practice (e.g., the teacher’s role and teaching methods).

After students complete Part II of Probe 1.1, for example, discuss their reactions to the questions. Note that the left side of the questionnaire on page 1-7 lists the assumptions underlying the skills approach; the right side, those underlying the NCTM Standards (the investigative approach). Underscore that in the conventional view, mathematics is simply stuff (unrelated information to be memorized by rote), children are merely stuffees (uninformed and helpless), and teaching is just a stuffing process (transmitting information and ensuring mastery by drill and practice).
Lesson 2. Use Investigation 1.2: *Totolospi* (pages 1-22 and 1-23 in the Student Guide) to illustrate how games can serve as a worthwhile (interesting and mathematically rich) task and as a basis for the investigative approach. This investigation can illustrate how a careful choice of worthwhile tasks can provide purposeful practice of basic skills, raise interesting issues, prompt new inquiry, and introduce new content. Playing the game *Totolospi* can be an entertaining way of practicing, for example, fraction comparisons intuitively (e.g., recognizing that \( \frac{2}{5} \) is less than one half and logically concluding that \( \frac{7}{9} \) must be greater than \( \frac{3}{7} \)). During the course of the game, players will encounter some comparisons that are hard to make intuitively (e.g., \( \frac{2}{7} \) versus \( \frac{5}{7} \)). This creates a need for more precise fraction-comparison methods. An instructor can encourage students to devise such methods and/or take this opportunity to discuss the rectangular cake-cutting analogy described on page 9-22 of Investigation 9.3 in chapter 9 of the Student Guide, a method that can naturally lead to the discovery of the formal common-denominator method.

Inevitably, some player comes up with a comparison such as \( \frac{2}{3} \) versus \( \frac{7}{6} \). This raises the issue of what fractions can represent (They can represent a quotient meaning as well as a part-of-a-whole meaning) and what dividing by zero means (Is it zero or undefined?). Often there is disagreement or uncertainty about the latter point. This can spark debate in which each side is encouraged to defend their position. Even if a class unanimously concludes that dividing by zero is undefined, an instructor can ask students to informally prove their claim. Some methods that can be used are relating \( 0 \div 7 \) and \( 7 \div 0 \), for instance, to a divvy-up meaning (no cookies shared fairly among seven children means each would get a share of zero vs. seven cookies shared fairly among no children means each nonexistent child’s share is zero ??!), using a calculator (e.g., keying, in \( 7 \div 0 \) = yields an error statement), relating examples to fractions (e.g., \( \frac{0}{7} \) = zero sevenths vs. \( \frac{7}{0} \) = seven noneths???), or relating examples to multiplication (e.g., \( 0 \div 7= ? \) can be thought of as \( x \times ? = 0 \), where the ? = 0, vs. \( 7 \div 0 = ? \) can be thought of as \( 0 \times ? = 7 \), where the ? has no satisfactory replacement).

*Totolospi* can also lead to a number of interesting probability issues, such as determining sample space, whether or not two outcomes are equally likely, empirical probability, and theoretical probability informally. One question (Questions for Reflection 6) can even involve students in informally exploring and determining expected values. The probability issues can also involve students in problem solving, conjecture-making and -testing, logical reasoning, and communicating. (Indeed, several class periods could be spent thoroughly exploring the mathematical issues related to playing *Totolospi.*)

Multiple-Activities Approach

Lesson 1. To help students examine the assumptions underlying to NCTM (1989, 1991) *Standards* and the investigative approach, an instructor could use the following series of activities drawn from the Student Guide and the Instructor’s Guide:

1. Use Investigation 1.1: *The Nature of Mathematics* (pages 1-9 and 1-10 in the Student Guide) to make the point that mathematics is not simply stuff (e.g., arithmetic facts and procedures as many people think) but is, at heart, an effort to find patterns in order to solve problems.

   To help students do Part II of Investigation 1.1, encourage them to decipher the symbols in the left-hand columns of each tablet first and then tackle the right-hand columns. Looking for patterns (inductive reasoning) often leads students to make both correct and incorrect conjectures. In either case, prompt them to test their conjectures (e.g., Does it explain all of the entries in the right-hand columns?).

   A common stumbling block is making sense of the last right-hand entry of Tablet 1. Many students induce the "multiples-of-nine" pattern of the column (9, 18, 27...) and deduce that the last entry must represent 63. Perhaps accustomed to base-ten thinking, many can’t see how the cuneiform translates into that number. Usually, though, someone conjectures that the stroke to the left must represent 60; the three strokes to the right, 3. Students can then be encouraged to check their conjecture by examining additional examples (Tablet 2) to see if it holds up.

   Note that Investigation 1.1 can serve as a basis for discussing the processes of mathematical inquiry (conjective-making and -testing, inductive and deductive reasoning, problem solving, and
communicating). This can be done informally as an instructor circulates around among the groups to check progress or afterward as part of a whole-class discussion.

2. Use Probe 1.2: Reflecting on Children's Mathematical Learning (page 1-12 in the Student Guide) and videotapes of children's impressive informal knowledge to counter conventional beliefs about children's learning (e.g., Children's mathematical thinking: Videotape workshops for educators by H. P. Ginsburg and colleagues and published by the Everyday Learning Corporation, Evanston, IL).

3. To highlight the differences between the skills approach and more meaningful approaches, an instructor can use Probe 1.C: Three Approaches to Teaching 4-Digit Subtraction in Base 4 (pages 20 to 25 in this guide). Part I simulates a skills approach to teaching a second-grade class the renaming algorithms. To help adult students appreciate how many children feel, this simulation involves learning base-four renaming procedures, which many, if not most, preservice teachers do not understand. Part I begins with prerequisite skills and illustrates that the skills approach is a hierarchical or bottom-up approach. To simulate this approach, briefly lecture students on a lesson and then have them complete the accompanying questions on their own. Discourage student-to-student communication. Students typically complete lessons 1 to 4 without much difficulty or understanding. To save time, an instructor can skip lessons 5 to 9 and get to the main demonstration: four-digit subtraction with renaming. After demonstrating the procedure and talking the class through it (e.g., for 3021 - 132, saying: "You can't take 2 from 1 so go to the fours column and rename the 2 one and the 1 in the ones column one-one . . . ."), assign seatwork Questions 1 to 6 on page 22. This typically creates confusion and despair. Adult students should readily see that rote memorization can carry children only so far before learning difficulties begin. To emphasize this point, ask students if they understand the prerequisite lessons and test their understanding by asking them to complete Question 2 under Questions for Reflection on page 22.

Part II of Probe 1.C can be used to simulate the conceptual and the investigative approach. Base-four blocks would be most helpful in completing this part of the probe (cubes = ones, longs = fours, flats = sixteen, and large cubes = sixty-fours). Completing lessons 1 to 5, which includes showing students a concrete model for adding multidigit numbers and explicitly linking this model to a written algorithm, simulates the conceptual approach. Lesson 6 can then be used to simulate the investigative approach. Announce that you are now going to play Backward Sixty-Four—that each player is going to start with a large cube (a 64) and subtract the numbers shown by a die roll (instead of adding the value shown by a die roll as was done in Sixty Four introduced earlier). As an example, roll the die and ask the students to use what they know to invent a concrete scoring procedure (e.g., "How would you concretely determine and represent your new score if the die roll came up three?"). After they have used a concrete scoring procedure for awhile, ask them to invent a written procedure that parallels it. Discuss with the class how the lesson simulates key features of the investigative approach and how it differs with the conceptual approach (e.g., students are encouraged to build on what they know to devise their own concrete model and written procedures).

Lesson 2. The aim of this lesson is to illustrate the investigative approach and its key features: purposeful, meaningful, and inquiry-based instruction. To achieve these aims, an instructor could use the following series of activities:

1. Use Investigation 1.3: The Role of Conflict in Learning (page 1-24 in the Student Guide) to illustrate that worthwhile tasks (in the form of mathematical problems) can provide a purposeful and engaging context for mathematical inquiry and learning. Working on Investigation 1.3 can create doubt and conflict, and these can motivate student exploration. In particular, it often leads to a debate on whether or not a square can be considered a rectangle. Working through Investigation 1.3 also naturally leads to a discussion of what determines the number of factors a number has and a recognition that it does not correlate with the size of a number. Furthermore, it can be used to make the point that teaching is, at heart, a "subversive activity"—that a teacher actively tries to promote doubt and conflict, because it is these things that cause students to reflect on what they know and advance their thinking.

2. Use Probe 1.3: Two Learning Exercises (page 1-26 in the Student Guide) to help students see that meaningful memorization (e.g., finding, relationships or connections) is more powerful than memorization by rote. Not only is the former
easier to do, it is more likely to be remembered (retained) and applied (transferred) to new learning tasks or problems.

3. Investigation 1.4: Zork Odd or Even (page 1-33 in the Student Guide) can further illustrate how a worthwhile task (in the form of a game in this case) can lead to inquiry. While playing Zork Odd and Even, inevitably the question is raised: Is zero an odd number, an even number, or neither? This can launch a class into making and testing conjectures, defining informally and/or formally what *even* means, and searching for ways to informally or more formally determine zero's status. Some methods students may suggest are using a number line (e.g., counting backwards from 10, every other number is even) or deductive reasoning (e.g., if an even number is one that is evenly divisible by two, then zero is even because $0 \div 2$ leaves no remainder).

**SAMPLE HOMEWORK ASSIGNMENT**

**Lesson 1**

Read: Unit 1•1 in chapter 1 of the Student Guide.

Study Group:

- *Questions to Check Understanding*: 1a to 1d, 2, and 3 (pages 27 and 28 of this guide).
- *Writing or Journal Assignment*: 6 (page 30).
- *Problem*: Picking Up Sides (page 31).
- *Bonus Problem*: Gaining Grains on pages 0-22 and 0-23 of the Student Guide.

Individual Journals: *Writing or Journal Assignment* 1 (page 29).

**Lesson 2**

Read: Unit 1•2 in chapter 1.

Study Group:

- *Questions to Check Understanding*: 1e-1o, 4, 5, and 6 (pages 27 to 29).
- *Writing or Journal Assignment*: 9 (page 30).
- *Problem*: Five-Color Grid (page 31).
- *Bonus Problem*: Rumor Mill City (page 31).

Individual Journals: *Writing or Journal Assignment* 14 (page 31).

**FOR FURTHER EXPLORATION**

**ADDITIONAL READER INQUIRIES**

**Probe 1.A** (page 18)

Identifying Approaches to Teaching Mathematics was designed to help adult students distinguish among the skills, conceptual, and problem-solving approaches and could be used in conjunction with Unit 1.1 (Lesson 1). Probe 1.4: The Investigative Approach (pages 1-28 to 1-32 of the Student Guide), which builds on Probe 1.A, could then be used to introduce the investigative approach.

**Probe 1.B** (page 19)

The Effects of the Traditional Skills Approach on Children's Disposition Toward Mathematics could serve to prompt a discussion about how this approach limits or even undermines children's mathematical power by fostering a negative disposition, by not providing opportunities to engage in mathematical inquiry (e.g., making and testing conjectures, problem solving, reasoning, and communicating), and not promoting meaningful learning. After students analyze the worksheet illustrated in Probe 1.B, ask them, for instance, "What purpose children would see in completing it and how this might impact on their motivation and beliefs."

**Probe 1.C** (pages 20 to 25)

Three Approaches to Teaching 4-Digit Subtraction in Base 4 can serve to compare and contrast the skills, conceptual, and problem-solving approaches. As noted earlier on page 16, it could also be modified to illustrate the investigative approach.

**QUESTIONS TO CONSIDER**

1. Compare and contrast how reading and mathematics are typically taught in the elementary school. What is the rationale for forming reading groups and for a teacher working closely with such groups? Is mathematics instruction typically conducted in the same manner? Why or why not?

2. (a) The teacher's edition of The Math Book From Hell recommends (text continued on page 26)
**Probe 1.A: Identifying Approaches to Teaching Mathematics**

The aim of this probe is to help you construct a framework for understanding and analyzing three different approaches to teaching mathematics: the skills, conceptual, and problem-solving approaches.

1. Summarized below are three approaches to teaching mathematics identified by Kuhs and Ball (1986, cited by A. Thompson, 1992). For each, indicate whether the approach described best fits the characterization of the skills approach, the conceptual approach, or the problem-solving approach.

   a. A **learner-focused approach** focuses on the learner's personal construction of mathematical knowledge. The teacher stimulates this construction by "posing interesting questions or situations for investigation, challenging students to think, and helping them uncover inadequacies in their own thinking" (A. Thompson, 1992, p. 136).

   b. A **content-focused approach with an emphasis on conceptual understanding** is driven by content but aims to help students learn the content in a meaningful fashion. A teacher points out the logic underlying procedures and the logical relationships of mathematical concepts.

   c. A **content-focused approach with emphasis on performance** is also content-driven but emphasizes student mastery of facts, rules, procedures, and formulas. Instruction is organized in a hierarchy from basic to complex and presented sequentially.

2. Cobb (1988) has suggested that teaching can be viewed as a continuum from teaching by imposition to teaching by negotiation.

   **Teaching by imposition** is based on the assumption that knowledge must be transmitted by teachers and absorbed by students. It entails a preset curriculum. A teacher serves as the expert, authoritatively dispensing information, evaluating the correctness of answers, and settling disagreements.

   **Teaching by negotiation** is based on the assumption that knowledge is constructed. In this approach, the material discussed follows from students’ questions or their felt need to know, and the teacher serves as a consultant—or even as a participant—helping children choose issues to explore, partaking in the solution of puzzling tasks, allowing the class to reach a consensus on acceptable answers, and arbitrating disagreements.

   **Teaching by mediation** involves a preset curriculum, but the teacher serves as a guide ensuring that instruction fits the readiness, interest and needs of the students and that students discover and understand prescribed concepts.

   In the graph below, identify which point (A, B, C, D, E, F, G, H, I, J, K, L, M, N, or O) best represent the skills approach. Do the same for the conceptual approach and the problem-solving approach.

3. Ausubel (1968) argued that teacher-directed instruction should help students internalize knowledge meaningfully, not absorbing it by rote. Where would his position fit in the graph above?
Probe 1.B: The Effects of the Traditional Skills Approach on Children’s Disposition Toward Mathematics

The aim of this probe is to encourage reflection about how the traditional skills approach undermines children’s disposition to learn mathematics.

1. A fourth-grade teacher tested her class on the same division worksheet every week. Dennis, now 21-years-old, still remembers the answers to the first row of expression ("Four, two, four, two, two, two, two") but does not recall the expressions themselves. What beliefs about mathematics do timed tests create and how might these beliefs undermine children’s disposition toward the discipline?

2. Consider the case of Jake, a bright 6-year-old kindergartner whose mother encouraged his efforts to compute two-digit sums such as 24 + 11 = 35. When a question came up in his class that involved a two-digit sum, Jake was the only one in the class able to solve it. Asked by his teacher how he solved the problem, Jake explained, "Twenty and ten is thirty, and the five (4 + 1) leftover makes thirty-five." Jake’s kindergartner teacher retorted angrily that he was doing the problem the wrong way, that he should not use such a method again, and that he should wait until first grade to be taught the "proper" way! (a) Why did Jake’s teacher respond the way she did? (b) What message does it send to Jake and the other children about using their informal mathematics? (c) How might his teacher’s attitude affect her students’ disposition toward mathematics?

3. In chapter 0 of the Student Guide (Part III of Probe 0.1), LeMar asked Miss Brill why we invert and multiply when dividing by a fraction? Jot down your reason for using this procedure. Share your explanation with your group or class. Analyze your responses. What does your analysis imply about how effectively school mathematics foster autonomy?

4. The traditional skills approach relies heavily on practice to help children master basic skills. Depicted to the right is a typical worksheet. (a) What purpose would children have for completing this assignment? (b) How could a teacher create a real need for practicing the basic addition combinations—that is, provide practice in a personally purposeful way to students?
Probe 1.C: Three Approaches to Teaching 4-Digit Subtraction in Base 4

The aim of this probe is to illustrate three approaches to teaching mathematics. Part I illustrates how the skills approach would teach 4-digit subtraction in base four. Part II illustrates the conceptual approach, and Part III describes the problem-solving approach. The topic of 4-digit subtraction in base four was chosen for two reasons. (a) Many preservice and in-service teachers have little or no understanding of the topic and, hence, it represents a genuine learning task for them. In effect, this task parallels the task primary-age children confront in learning 4-digit subtraction with renaming (borrowing) in our familiar base-ten system and, thus, should help many readers better appreciate what it feels like to be in the shoes of children. (b) Moreover, learning about base-four arithmetic can deepen your understanding of our base-ten number system and arithmetic procedures.

Part I: The Skills Approach

To teach a complex skill such as base-four 4-digit subtraction with renaming, the skills approach would first teach prerequisite basic skills (Lessons 1 to 9). To save time, you may wish to complete Lessons 1 to 5 and then skip directly to Lesson 10. As you go through these lessons, consider the following questions: (a) Do you understand what you are doing? (b) How do you feel? (c) How is your experience analogous to that of primary-grade children learning base-ten, place-value skills such as multidigit subtraction in a traditional classroom?

Lesson 1: Master the first prerequisite skill—base-four subtraction facts. Shown below are the basic (single-digit) base-four subtraction combinations. (We will assume that you already understand, for instance, that two take-away one is one.) Write the answer to any incomplete expression. Then study any fact that you do not have memorized.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 \\
-0 & -0 & -0 & -1 & -1 & -1 & -2 & -2 \\
1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\
\end{array}
\]

Lesson 2: Master second prerequisite skill—place-value with two-digit numbers.

1. a. Fill in the blanks as shown for Item 1.

1. \( \frac{1}{\text{four}} \) \( \frac{2}{\text{ones}} \)

2. \( \frac{\text{fours}}{\text{ones}} \)

3. \( \frac{\text{four}}{\text{ones}} \)

b. Write a number for each block display as shown for Item 1 below.

1. 12 

2. ___ 

3. ___
Lesson 3 and 4: Master the third and fourth prerequisite skills—place-value representation of three- and four-digit base-four numbers.

a. One bundle of 16, one bundle of four, and two single sticks is written 112. Now for the second display fill in the blanks:

1. 

2. 

b. Write a number for each block display.

<table>
<thead>
<tr>
<th>64s</th>
<th>16s</th>
<th>4s</th>
<th>1s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>64s</th>
<th>16s</th>
<th>4s</th>
<th>1s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Lesson 5: Master fifth prerequisite skill (two-digit subtraction without renaming).

Subtract the following:

<table>
<thead>
<tr>
<th>a. 32</th>
<th>b. 21</th>
<th>c. 23</th>
<th>d. 33</th>
<th>e. 20</th>
<th>f. 22</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>-11</td>
<td>-10</td>
<td>-12</td>
<td>-10</td>
<td>-11</td>
</tr>
</tbody>
</table>

Lesson 6: Master the sixth prerequisite skill (two-digit subtraction with renaming). In Item a, note that because you cannot subtract 3 from 2, you must rename one four four ones. That is, cross out the two in the fours column and write 1 (four), and then show the four ones by writing a 1 above and to the left of the 2 in the ones column. Use this procedure to complete items b through d.

<table>
<thead>
<tr>
<th>a. 17 12</th>
<th>b. 21</th>
<th>c. 32</th>
<th>d. 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 1 3 3</td>
<td>-12</td>
<td>-13</td>
<td>-11</td>
</tr>
</tbody>
</table>
Lessons 7, 8, & 9: Master the seventh prerequisite skill (three-digit subtraction without renaming), the eighth prerequisite skill (three-digit subtraction with renaming), and the ninth prerequisite skill (four-digit subtraction without renaming).

<table>
<thead>
<tr>
<th>123</th>
<th>132</th>
<th>223</th>
<th>323</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 12</td>
<td>-112</td>
<td>-13</td>
<td>-120</td>
</tr>
<tr>
<td>121</td>
<td>132</td>
<td>120</td>
<td>101</td>
</tr>
<tr>
<td>- 12</td>
<td>- 23</td>
<td>- 13</td>
<td>- 12</td>
</tr>
<tr>
<td>1231</td>
<td>1312</td>
<td>2311</td>
<td>3201</td>
</tr>
<tr>
<td>- 121</td>
<td>- 210</td>
<td>- 201</td>
<td>- 100</td>
</tr>
</tbody>
</table>

Lesson 10: Four-digit subtraction with renaming.

Step 1: Start with the ones digit, rename if necessary, and subtract. In the example to the right, 2 cannot be taken from 1, so the 2 in the fours column is renamed as 1 four and 4 ones. This now makes five ones, which is represented in the ones column as 11 (see Figure A). Five ones minus 2 ones is three ones.

A. 3 0 \underline{1} 1
   - 1 3 2
   \underline{3}
   B. 2 1 1
      - 1 3 2
      \underline{2 3}

Step 2: Move to the next column to the left (the fours column), rename if necessary, and subtract. In Figure B to the right, note that 3 fours cannot be taken from 1 four. It is not possible to rename a sixteen, so the 3 sixty-fours must be renamed as 2 sixty-fours and 4 sixteens. In Figure C, the 4 sixteens have been renamed as 3 sixteens and 4 fours, which results in a total of 5 fours, making it possible to complete the subtraction of the four columns.

C. 2 \underline{3} \underline{1} 1
   - 1 3 2
   \underline{2 3}
   D. 2 \underline{3} \underline{1} 1
      - 1 3 2
      \underline{2 2 2 3}

Step 3 & 4: Repeat Step 2 with the sixteens column and sixty-fours column (see Figure D).

Now practice this procedure with the following examples. (If this seems like an overwhelming task, that's the point. Without understanding, it is unlikely that most people would complete the following assignment successfully.)

1. 1332
   - 203
2. 1221
   -1012
3. 1030
   -1012
4. 1202
   -123
5. 1200
   -122
6. 1102
   -113

Questions for Reflection

1. Why do you suppose many teachers use the skills approach to teach mathematics?
2. Compare your answers to Lessons 1 to 10 with your group or class. (a) On what lessons were you successful (got all the correct answers). (b) How many of your group or class were also successful on each of the lessons? (c) Does successfully completing a lesson necessarily mean you understand the underlying concepts of the lesson? For example, if you successfully completed Lessons 3 and 4 on page 21, can you answer the following questions that require an understanding of base four: (i) The number 155 in base ten would be written how in base four? and (ii) The base-four numeral 1132 represents what base-ten numeral? (d) How many people in your group or class completed Lessons 3 and 4 successfully but were unsuccessful on the items i and ii? (e) What are the implications of these results?
3. (a) Was there a point where you became so lost that you could no longer perform an assigned task successfully? If so, at what lesson did this happen? (b) Where did others in your group or class meet their Waterloo (if they did)?
Part II: The Conceptual Approach

For most students, the skills approach of Part I is fruitless and frustrating. The conceptual approach is more likely to achieve our ultimate goal of learning how to subtract 4-digit numbers in base four. You will need base-four blocks for Part II. (Multibase blocks are commercially available from Tools for Teaching. See the footnote at the bottom of page 147 of this guide for the address for this firm and other ordering information.) If base-four blocks are not available, use interlocking blocks, paper cutouts representing the blocks, or drawings of the blocks. Work in groups of about four to complete Part II.

Lesson 1: Counting in base four. The planet Zork is inhabited by beings who have two fingers on each hand. Consequently, the counting system that evolved on Zork is different from that on our planet earth. For example, to count a collection such as ••••, Zorkians would count one, two, three, zork instead of one, two, three, four. Zorkians do not use our counting term four or any of our counting terms beyond four such as five or twelve. (1) What do you think comes after zork when Zorkians count? (2) What counting sequence do you think a Zorkian would use to count a collection of 15 items? (3) The Zorkian count term for 16 is super zork; the term for 64 is super-duper zork. What do you think are the counting terms in between?

Lesson 2: Grouping by fours. Play Sixty-Four using base-four blocks. On their turn, players roll a die with 0 to 5 dots and collect the number of cubes indicated by the roll. Now one way to keep score would be to go on collecting cubes. However, this would mean collecting many cubes, which would be hard to count accurately or which might use up the supply of cubes (and upset your opponents). To make keeping track of large numbers of things easier, the Zorkians grouped by fours. That is, whenever players got four of something, they traded it in for a larger unit. More specifically, whenever players collect four cubes, they trade them in for a long (a group of four, which Zorkians call a zork). Four longs can be traded in for a flat (a group of 16, which Zorkians call super zork); four flats can be traded in for a big cube (a group of 64, which Zorkians call super-duper zork). The first player to collect a super-duper zork is the winner.

Lesson 3: Introduction to base-four place value. Play Sixty-Four again, but this time keep a running score of the blocks you collect in a chart like that below.

<table>
<thead>
<tr>
<th>big cube</th>
<th>flat</th>
<th>long</th>
<th>cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>64s</td>
<td>16s</td>
<td>4s</td>
<td>1s</td>
</tr>
</tbody>
</table>

(Turn 1: Rolled a 3, collected 3 cubes. Score = 3 units.)
(Turn 2: Rolled another 3, collected 3 cubes for a total of 6 cubes.)
(Four cubes traded for 1 long. Score = 1 group of four and 2 units.)
(Turn 3: Rolled a 2, collected 2 cubes for a total of 1 long and 4 cubes.)
(Four cubes traded for another long. Score = 2 groups of four and no units.)
(Turn 4: Rolled a 5, collected a long and a cube.)
(Total Score = 3 groups of four and 1 unit.)

Lesson 4: Base-four numeration. To figure out the Zorkian number for a collection, consider how many groups of sixty-four, sixteen, and four can be made. For example in the case of seven items, the largest group that can be made is a group of four, which would leave three ungrouped items. This would be written in Zorkian as 134. With 21 items, it is possible to make a group of 16 and a group of 4, leaving one ungrouped item. This would be written in Zorkian as 1114. Seventy-five items could form a group of 64, no groups of 16, two groups of 4, and
Probe 1.C continued

three ungrouped items. This would be represented \(1023_4\). These examples are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>sixty-fours (groups of 64)</th>
<th>sixteens (groups of 16)</th>
<th>fours (groups of 4)</th>
<th>ones (ungrouped items)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>75</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that without the zero, \(1023_4\) would be confused with \(1234\), which represents 1 group of sixteen, 2 groups of four, and 3 ungrouped items (or the number 27 in base 10). Now try your hand at the following questions:

1. Earthlings with the following ages would be how old on the planet Zork? Indicate the equivalent Zorkian age of each in terms of a (a) Zorkian counting term, (b) Zork (base-four) blocks, and (c) Zorkian (base-four) written number (numeral).

<table>
<thead>
<tr>
<th>Base-ten age</th>
<th>Zork age</th>
</tr>
</thead>
<tbody>
<tr>
<td>count term</td>
<td>numeral</td>
</tr>
<tr>
<td>four</td>
<td>4</td>
</tr>
<tr>
<td>six</td>
<td>6</td>
</tr>
<tr>
<td>ten</td>
<td>10</td>
</tr>
<tr>
<td>twelve</td>
<td>12</td>
</tr>
<tr>
<td>sixteen</td>
<td>16</td>
</tr>
<tr>
<td>twenty-four</td>
<td>24</td>
</tr>
<tr>
<td>thirty-seven</td>
<td>37</td>
</tr>
<tr>
<td>fifty nine</td>
<td>59</td>
</tr>
<tr>
<td>sixty four</td>
<td>64</td>
</tr>
</tbody>
</table>

2. Determine the larger of the two numbers in each of the following cases:
   a. \(13_4\) or \(8_{10}\)
   b. \(20_4\) or \(7_{10}\)
   c. \(32_4\) or \(14_{10}\)
   d. \(102_4\) or \(19_{10}\)
   e. \(220_4\) or \(40_{10}\)
   f. \(1101_4\) or \(82_{10}\)

Lesson 5: Linking concrete and symbolic multidigit addition.

1. If four and six blocks were combined, we could represent this situation by writing \(4 + 6 = 10\).
   a. Model the situation with base-four (Zork) blocks.
   b. Write an equation in Zorkian (base-four) notation for the model you just made.

2. Use Zork blocks to figure out the answer to the following expression: \(133_4 + 21_4\)

Lesson 6: Play Backward Sixty-Four. Start with a large cube (super-duper zork) and subtract the value of each roll. Keep a written score using the format shown to the right.

Lesson 7: Linking concrete and the symbolic multidigit subtraction. Using the place-value mat below, illustrate how the starting amount \(2,032_4\) would be represented with base-
Probe 1.C continued

four blocks. Now consider how to take away 1,133. Take away like blocks starting with smallest blocks. Can you take 3 cubes from the starting amount’s 2 cubes? What must you do to have enough cubes to remove 3? Consider how this would be represented symbolically. Continue the process of subtracting with the blocks. Indicate how each step in the concrete procedure illustrates a step in the written algorithm.

Questions for Reflection

1. (a) What are the advantages of the conceptual approach over the skills approach? (b) What are the disadvantages?

2. Do the advantages outweigh the disadvantages?

Part III: The Problem-Solving Approach

The problem-solving approach would begin with a problem such as the one below:

**A Scoring Procedure for Sixty-Four.** In *Sixty-Four*, players roll a die to determine how many points they get on their turn. The first player to get 64 points wins. Use blocks to keep score. Cubes represent one point each. To make counting the points easier, when you get four cubes, trade them in for a long (a group of four). When you get four longs, trade them for a flat (a group of sixteen). And when you get four flats, trade them in for a large cube (a group of sixty-four). The first player to get a cube wins. Keep a written record of each of your scores in case there is not enough time to complete the game today.

In effect, students are challenged with the genuine problem of inventing an efficient written system for keeping a running score. In fact, one study (Pengally, 1988) asked first graders to solve a similar problem (*A Scoring Procedure for One Hundred*) using base-ten blocks—grouping 10 cubes (ones) into a long (ten) and 10 longs (tens) into a flat (hundred). Initially, children described their scores in words—a procedure that was time consuming and difficult to interpret readily. In time, children invented shorthands with numbers. By the end of the school year, some children had invented efficient written procedures that resembled the standard renaming (carrying) procedure. In effect, these children had *re-invented* the elegant algorithm that adults depend on to do written multidigit addition.

Once students had invented an efficient procedure for performing written multidigit addition, they could be challenged to do the same for multidigit subtraction. For example, they could be asked to keep a written running score for *Backward Sixty-Four* (for base four) or *Backward One Hundred* (for base ten).

Questions for Reflection

1. (a) What are the advantages of the problem-solving approach? (b) What are its disadvantages?

2. Consider how the conceptual and the problem-solving approach could be blended to take advantage of the best features of each approach.
using star charts to motivate students to memorize the basic addition facts (sums to 18). What are the drawbacks of this technique? (b) A researcher asked elementary students to tutor younger children (Garbarino, 1975). One group of tutors was promised a reward if their tutees learned the material. Another group was not promised anything for helping. What would you predict was the outcome of this study: the group promised rewards was more successful helping their tutees to learn, the two groups were equally successful, or the group promised nothing was more successful? Why?

SUGGESTED ACTIVITIES

1. Examine a textbook at a grade level of your choice. Evaluate its content and instructional suggestions in terms of WHAT THE NCTM CURRICULUM STANDARDS SAY (pages 1-3 and 1-4 of the Student Guide). Note which items that are supposed to receive increased attention, in fact, do and which do not. Do the same for the items that are supposed to receive decreased attention.

2. Visit an elementary mathematics classroom for a week. Write a report of your analysis of the instruction. (a) Indicate whether the instruction was predominantly the skills, conceptual, investigative, or problem-solving approach. Justify your classification. (b) What impact did the instruction seem to have on the children’s mathematical power, including their disposition toward mathematics, their ability to engage in mathematical inquiry, and their understanding of mathematics? (c) How could the instruction be modified to be more consistent with the investigative approach? (d) Explain how your modifications would enhance children’s mathematical power.

3. Using the information at the bottom of page 1-3 in the Student Guide, write, call, or e-mail the NCTM to obtain (a) a membership application, (b) a catalogue of NCTM publications and products, (c) information about the NCTM Standards, (d) a subscription to JRME Online, or (e) information regarding a question that you have about mathematics teaching.

4. Visit the NCTM web page (http://www.nctm.org). Write a report summarizing how it could be helpful to other preservice or in-service teachers.

5. Choose a topic at a grade level you plan to teach and design a lesson based on the investigative approach. Consider how you can make the instruction of this topic purposeful, inquiry-based, and meaningful. More specifically, specify the content goals of the lesson, including what new skills or concepts would be introduced and what skills and concepts would be reviewed or practiced. Indicate also any process goals (e.g., introducing the problem-solving strategy of drawing a picture, practicing logical reasoning, or posing and testing conjectures). Describe the project, problem, game, activity, or real classroom situation that would provide a context and purpose for teaching the lesson content. Spell out how the project, problem, game, activity, science experiment, or real classroom situation might be interesting to children and how it would create doubt, conflict, or otherwise motivate students to learn or use the content prescribed in the content goals. Explain how the content questions or issues raised by the lesson would be explored in an inquiry-based and meaningful way (e.g., how instruction would build on children’s existing informal or formal knowledge).

6. (a) Use the Annenberg/CPB Math and Science Project web site (http://www.learner.org/k12), MathFinder CD (see page 1-38 in the Student Guide), or other technology-based resource to find a lesson that could serve as the basis for an investigative approach to a topic of your choice. Print out the lesson idea. (b) Write up a lesson plan based on the print out. Indicate the appropriate grade level and any modifications you have made to the lesson. (c) Justify how the lesson meets the criteria of the investigative approach.

7. (a) Select a topic in mathematical history and research it using books and/or the world wide web (see, e.g., MacTutor History of Mathematics Archive at http://www-groups.dcs.st-and.ac.uk/~history/index.html). (b) Write a report that discusses how the topic could be used in an integrated lesson involving mathematics and social studies.

8. (a) Read Mathematics and Science: An Adventure in Postage Stamps by William L. Schaaf © 1978 by the NCTM. (b) Choose a mathematical achievement or a mathematician and create a display of relevant stamps and commentary.
9.† Videotape or audiotape a mathematics lesson by another teacher or yourself. Analyze the recorded lesson in terms of the following questions: (a) Did the teacher allow children to explain how they solved a problem without passing judgment about the solution? (b) Did the teacher ask follow-up questions in order to better interpret their thinking or elaborate on their ideas? (c) Did the teacher allow children to self-correct incorrect answers or ask questions to encourage them to rethink their answers? (d) Did the teacher ask children to justify their strategy or solution? (e) Did the teacher seek out different strategies, methods, or procedures or solutions and have the children compare and contrast them? (f) How long did the teacher wait for children to respond to questions before giving them the answer? (g) Did the teacher encourage the students to question and challenge each other?

10. Visit a classroom during mathematics instruction for a week. Describe the teacher’s attempts to motivate inquiry and learning by seizing opportunities that created cognitive conflict and debate. Describe what opportunities were lost and how you might have handled them differently to better motivate inquiry and learning.

11. Visit JRME Online (the electronic version of the Journal for Research in Mathematics Education) at the following URL: http://www.nctm.org/jrme/. Subscription information can be obtained from the NCTM web page or by calling the NCTM (see page 1-3 of the Student Guide). Choose a research article that is relevant to your interest. Write a report on the educational implications of the research findings.

**HOMEWORK OR ASSESSMENT**

**QUESTIONS TO CHECK UNDERSTANDING**

1. Circle the letter of any of the following statements that—according to the Student Guide—is true.

   a. Primary-level children, particularly, bring little or no mathematical knowledge to school and, hence, are essentially helpless to solve problems on their own.

   b. Systematic errors, such as counting " . . . twenty-eight, twenty-nine, twenty-ten" are clear evidence that children actively construct knowledge.

   c. Research suggests that children have limited attention spans and can take in new information for only a short period of time.

   d. Using rewards is essential, because children come to school with little or no interest in exploring mathematics.

   e. Teachers should encourage students to invent their own procedures as well as re-discover procedures.

   f. Games are useful for practicing skills but not for teaching concepts.

   g. A useful instructional technique is to begin lessons with a question or a problem.

   h. A goal of the investigative approach is routine expertise.

   i. Unlike the skills approach, the investigative approach focuses on the processes of mathematical inquiry, as well as the learning of content.

   j. The investigative approach focuses on the meaningful memorization of facts, rules, procedures, and formulas.

   k. Like a whole-language approach to reading, the investigative approach is a top-down approach—begins with a relatively complicated task and teaches basic skills as they are needed.

   l. Unlike planning a lesson based on the problem-solving approach, planning a lesson based on the investigative approach begins by considering content objectives as well as process objectives.

   m. With the investigative approach, a teacher would pose a problem and show the students how it could be solved using manipulatives.

   n. A teacher should avoid creating cognitive conflict.

   o. In a world that increasingly depends on information and technology, achieving au-
2. Preservice teachers were asked to complete the phrase, "To me math is . . .". For each response below, identify which of the following conceptions of mathematics it best fits: mathematics as skills, mathematics as a network of skills and concepts, or mathematics as a way of thinking. Briefly justify your choice.

a. . . . learning basic procedures that one will use throughout his/her entire life.

b. . . . a way to solve problems.

c. . . . a subject that takes a great deal of . . . memorization [by rote]. One must perfect the [procedures] being taught in order to achieve the right answers.

d. . . . a way of looking at things and making sense of them. By finding patterns, other answers can be found. Math is an essential tool in daily life.

e. . . . understanding procedures so that they can be effectively applied to both familiar and unfamiliar everyday situations.

3. In comparison to memorizing mathematics by rote, meaningful memorization of related mathematical facts and ideas is (circle the letter of any of the following statements that are true according to the Student Guide):

a. more difficult to achieve
b. more likely to be remembered
c. more likely to make learning new material easier
d. more likely to be useful when solving problems
e. less likely to require review
f. less time consuming in the long run
g. more likely to foster a positive disposition toward mathematics

d. . . . learning basic procedures that one will use throughout his/her entire life.

e. . . . a way to solve problems.

c. . . . understanding procedures so that they can be effectively applied to both familiar and unfamiliar everyday situations.

4. Table 1.1 below compares the skills approach, the conceptual approach, and the problem-solving approach in terms of the following categories: aims, focus, roles of the teacher and the students, organizing principle, and key instructional methods. If a fourth column, titled the Investigative Approach, were added to the table, what would be an appropriate description for each category?

<table>
<thead>
<tr>
<th>Table 1.1: Three Approaches to Teaching Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Skills Approach</strong></td>
</tr>
<tr>
<td><strong>Aim:</strong> To foster mastery of basic skills—the memorization of arithmetic and geometric facts, rules, formulas, and procedures by rote</td>
</tr>
<tr>
<td><strong>Focus:</strong> Procedural content (e.g., how to divide with fractions)</td>
</tr>
<tr>
<td><strong>Roles:</strong> Teacher directed; students passive and dependent</td>
</tr>
<tr>
<td><strong>Organizing principle:</strong> Sequential instruction based on a hierarchy of skills—build from simplest to most complex skill (bottom-up)</td>
</tr>
</tbody>
</table>
| **Methods:**  
- Direct instruction; primarily teacher talk  
- Practice with an emphasis on written worksheets  
- Little or no use of manipulatives | **Methods:**  
- Direct instruction supplemented by guided discovery-learning  
- Teacher uses, e.g., meaningful analogies and concrete models to explain procedures  
- Teacher demonstrates concrete model; children imitate manipulative procedure | **Methods:**  
- Student discussion of problems, solution strategies, and answers  
- Content instruction done incidentally as needed  
- Children use manipulatives to invent their own concrete procedures or to justify their conclusions |
5. Eddie did not understand the multidigit addition renaming procedure. For example, for 38 + 27, he got an answer of 515. Circle the one strategy below that best illustrates the investigative approach recommended by the Student Guide?

a. Provide no feedback or correction. Allow him to continue making the error, because sooner or later he will discover for himself what he is doing wrong.

b. Provide feedback and correction. Note that the answer is incorrect and point out to him how to do the procedure correctly using base-ten blocks.

c. Do not provide feedback but create cognitive conflict. Ask him if his answer makes sense or have him compare his answer to others.

d. Do not provide feedback but provide correction. Simply show him how to do the procedure correctly using base-ten blocks—thus, accentuating the positive.

6. In Mr. Yant’s accelerated Math 8 class, Alissa Weinberg devised a shortcut for the school-taught method of rationalizing the denominator of a fraction such as $\frac{12}{\sqrt{3} - 6}$. In the formal method (Method A in Figure 1.1 below), the first step involves multiplying both the numerator and the denominator by $\sqrt{3} + 6$. The second step involves determining the products in each case. Alissa suggested not finding the product of the terms in the numerator (Step 2 in Method B in Figure 1.1) until the denominator was simplified, which simplifies the reducing process (Step 3 in Method B). This eliminated Step 4 in the formal process—factoring to reduce. This shortcut was a new idea for Mr. Yant. According to the Student Guide, how could Mr. Yant use this opportunity to build a mathematical community?

**WRITING OR JOURNAL ASSIGNMENTS**

1. (a) Answer the following question: Why do we study mathematics? (b) Describe an example of how thinking mathematically increased your personal power or gave you an advantage.

2. A fellow kindergarten teacher welcomes you, a first-year teacher, with the following advice, “Children this age are adorable, but they have short attention spans. After a few minutes, they get all fidgety and their minds begin to wander. That’s why I plan all of my activities for no more than 10 minutes.” Evaluate this advice in light of what was discussed in chapter 1 of the Student Guide.

3. After a workshop on teaching mathematical problem solving, one of your colleagues complains, “I can’t imagine asking my students to solve problems. They’re so helpless they wouldn’t know where to start. I have to tell them everything.” Evaluate this teacher’s complaint in light of the facts and arguments laid out in the Student Guide.

4. A behavioral psychologist reported that one day he come home from work to find that his son had cleaned up his room. Wishing to encourage such behavior, the psychologist re-
warded the boy with a dollar. The next day, the psychologist came home and was disappointed to find that his son had not picked up his room. Moreover, when he asked his son for a hug, the boy stuck out his hand—a gesture that he wanted to be paid first. What does the vignette illustrate about the use of rewards?

5. The "spiral curriculum" refers to an instructional program that introduces a concept at a basic level and then repeatedly revisits the concept at more and more sophisticated level—each time building on what was taught earlier. The elementary mathematics curriculum used in the traditional skills approach has been described as a "spiral curriculum with a radius of almost zero." (a) What does this mean? (b) Such a curriculum has what effect on students' disposition toward mathematics? Why?

6. In debating changes in the mathematics curriculum, one of your colleagues argues, "Meaningful instruction simply isn't realistic because it takes too long. I barely get through our textbook now. If I took the time to help my students understand the material, I'd never finish the book." Evaluate your colleague's argument in light of what was discussed in the Student Guide.

7. At Parents' Night, you proudly outline to the parents of your class the main goals of your mathematics program—solving problems, reasoning, communicating, and understanding (making connections). A parent stands up and announces, "I read an article that said it was time that we got back to teaching the basics—memorizing arithmetic facts and procedures—because too many kids can't do simple arithmetic. Why are you teaching my kid all this other stuff he won't need rather than the basic skills he will need?" Develop an argument that will convince parents that the investigative approach, which focuses on mathematical processes and understanding, is a better way of teaching mathematics than the skills approach. (Note that such an argument should also be useful with colleagues and supervisors who are skeptical of change.) Share your arguments with your group or class. Afterward, refine your argument.

8. (a) Miss Brill described her colleagues to her parents. "Mrs. MeChokemchild uses the traditional skills approach; Ms. Socrates uses the investigative approach," she noted. Her parents asked for clarification. How might Miss Brill characterize and contrast these two approaches? (b) Compare and contrast the investigative approach, the conceptual approach, and the problem-solving approach.

9. In response to the "To Me Math Is..." activity (see page 0-1 of the Student Guide), one student wrote, "I like math because it is clear-cut. There's no debating what's right or wrong. There is one right way to do things." Another wrote, "Math is a subject of correct answers—no negotiating." (a) Is such a view of mathematics accurate? Why or why not? (b) According to the description of the skills, the conceptual, the investigative, and the problem-solving approaches in the text, which approach or approaches would most likely foster such beliefs? Why? (c) Which would be the least likely? Why?

10. To find the area of a rectangle, 4\(\frac{1}{2}\)-inches by 3\(\frac{1}{4}\)-inches, a student proposed: Multiply the 4 and the 3—that's 12. Then multiply the \(\frac{1}{2}\) and the \(\frac{1}{4}\)—that's \(\frac{1}{8}\). So the answer is 12\(\frac{1}{8}\). In the investigative approach, how would a teacher respond to this conjecture?

11. Mathematics educators differ about how much a teacher should intervene. Some argue for a laissez faire (hands-off) approach where a teacher accepts a class' consensus, even if it is incorrect. Advocates for this approach note that it maximizes the development of autonomy and a genuine mathematical community. Besides understanding cannot be imposed anyway, and children can always arrive at the correct conclusion later when they are developmentally more ready. Other mathematics educators argue for a more authoritative approach where a teacher allows open debate but then challenges incorrect conclusions. Advocates of this approach note that peer dialogue may or may not be productive and a teacher is responsible for quality control. Moreover, practical considerations such as limited time make it necessary for a teacher to step in when a class arrives at an incorrect conclusion. Which position, if either, does the Student Guide recommend? Justify your answer.
12. (a) Mr. Watkins introduced multidigit subtraction with renaming (borrowing) by teaching them a small-group game called Sales Race in which the students pretended to be sales people. Each player started with an inventory of 250 units of merchandise. To determine how many items they sold on their turn, players rolled two ten-sided dice and arranged the dice to make the largest number possible. The number of sales was then subtracted from their inventory. The first player to reduce his or her inventory to 0 was the winner. To illustrate the scoring, Mr. Watkins had a student roll the dice. Natalie rolled a 3 and 6 and arranged them to form 63. The class then considered how to take 63 away from 250. Some students concluded that it couldn't be done, others argued that the answer was 107, and many favored an answer of 213 (proceeding right to left, subtract smaller digits from larger: 0 - 3 = 3, 6 - 5 = 1 and 2 - 0 = 2). This lead to a debate about the different strategies and answers proposed. Ultimately, the majority won out and the class concluded that 213 was correct. According to the advice presented in the Student Guide, how should Mr. Watkins respond to the class' consensus? Should he intervene or not? Why or why not? (b) After a lively discussion, Ms. Ball’s third-grade class concluded (incorrectly) that positive infinity minus negative infinity is zero. Should she intervene or not? Why or why not?

13. Asked to look for odd-even patterns for multiplication, a group excitedly announced to their teacher, "Whenever you multiply with an odd number the answer is even." According to the advice presented in the Student Guide, how should the teacher respond to this incorrect conclusion?

14. (a) What letter (A, B, C, . . . or Y) from the chart on page 1-32 of the Student Guide best represents your own elementary mathematics instruction? Your high school mathematics instruction? Your college-level mathematics instruction? Briefly justify. (b) Which of the 25 types of teaching approaches represented in the chart would you feel comfortable assuming today? Why? (c) Which type or types do you consider your ideal approach? Why? (d) What obstacles, if any, might hinder achieving your ideal?

15. On page 58 of Everybody Counts, the National Research Council (1989) noted: “No one can teach mathematics.” In light of the discussion on learning in Chapter 1 of the Student Guide, what do you suppose this statement means?

PROBLEMS

■ Picking Up Sides (◆ 5-8)

Thinking mathematically can be advantageous in many everyday situations. Consider the following situation: The gym teacher chose Adli and Bernard as captains. The captains alternately chose boys for their teams. Note that going first meant getting the better picks overall (e.g., the best player, the third best player, the fifth best player, the seventh best player, and so on, rather than the second best player, the fourth best player, the sixth best player, the eighth best player and so on). To be fair, the coach had the captains choose a number between 1 and 10 and whoever was closer would pick first. Unlike Bernard, Adli analyzed the task in quantitative terms and devised a strategy to maximize his chances of winning and choosing first. (a) What was Adli’s strategy? (b) By thinking mathematically, the gym teacher could have devised a fairer way for the captains to choose their players. What would be a fairer method than alternating choices? Why?

■ Rumor Mill City (◆ 5-8)

Beth Bloom’s boyfriend bought a diamond engagement ring as a surprise for New Year’s Eve, a mere 19 hours away. Although he swore Mr. Gossip to secrecy, within 1 hour the jeweler had told two others the secret anyway. Within 2 hours, these two people had each told two more of their friends and so forth. Rumor Mill City has a population of 524,300. Assume that Beth was the last person in the city to hear the news about her engagement, each person told only two others the news, and no one heard the news from another more than once. Was Beth surprised or not by her boyfriend’s announcement? Justify your answer.

■ Five-Color Grid (◆ 3-8)

Astoria wanted to make a patchwork table cloth using squares of the following colors: red, orange, yellow, green, and blue. She wanted each row, column, and diagonal to have five different colors. How could she sew the 25 squares of material together to accomplish this?
ANSWER KEY for Student Guide

Key for Investigation 1.2: Questions for Reflection (page 1-23)

1. (c) Some students may claim that there is a one in three chance of getting a painted and unpainted side up, because there are three different outcomes. Others may argue that there is a one in four chance, because there were two ways of getting a painted (p) and unpainted (u) stick to come up. If the former counter that p-u or u-p are equal and that there is no way to distinguish between these two possibilities, encourage students to evaluate this conjecture or to settle the debate by determining the empirical probability of each outcome—by actually playing the game a number of times and compiling the results.

2. Lanny’s group made a common error among children; they assumed that each of the three outcomes (two painted sides, two unpainted sides, and a painted and an unpainted side) was equally likely.

3. (a) 12 spaces ÷ 3 spaces max/turn = 4 turns. (b) \( p = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256} \). (c) \( \frac{1}{4096} \).

5. (a) It could take an infinite number of turns because there is always the possibility, however small, that a player will not roll the exact number to get out. If students answer otherwise, remind them that a player must get the exact number to finish.

6. (a) Akemi’s group\(^3\) set up the table below. They reasoned that, on any one turn, a player should average 7 (total points) ÷ 4 (possibilities) or 1.75 points. They further reasoned that it would take 12 points to finish. Thus, a player could, on average, expect to take 12 ÷ 1.75 or 6.85714… (approximately 7) turns to finish. This is an intuitive equivalent of a formal solution based on an expected value. The probability of obtaining 3 points (a RR) is \( \frac{1}{4} \); 2 points (a WW), \( \frac{1}{4} \); and 1 point (a WR or RW), \( \frac{2}{4} \). Thus the expected value is \( 3 \left( \frac{1}{4} \right) + 2 \left( \frac{1}{4} \right) + 1 \left( \frac{2}{4} \right) = \frac{3}{4} + \frac{2}{4} + \frac{2}{4} = \frac{7}{4} = 1.75 \). Again, dividing 12 by 1.75 gives the expected number of turns to finish the game.

Key for Probe 1.3 (page 1-26)

Part I

Brightness is represented by the symbols for the two brightest heavenly bodies: the sun and the moon. Note that dawn is represented by the symbol for the sun with a stroke underneath it to represent the horizon, east is represented by a tree with the (rising) sun behind it, and month is represented by the same symbol for moon. (The Chinese used a lunar calendar.) Note that extremely hot is represented by two fire symbols. A forest is shown as two trees; a dense forest, as three trees. Good is the combination of the symbols for woman and child; peace, the combination of roof and woman. Home is the combination of roof and pig, which may have stemmed from peasants once keeping pigs in their home for warmth.

Questions for Reflection

2 to 4. The conventional wisdom that an approach that focuses on memorization by rote is a more efficient way to teach than one that focuses on meaningful memorization or learning is wrong for three reasons.

- Meaningful learning is typically easier than learning by rote (Sawyer, 1964). It is often assumed that memorizing information by rote is easier than meaningful learning—particularly for children with learning difficulties. Consider, though, Part I. Memorizing Chinese characters one by one by rote would require considerable time and effort. Meaningful learning—taking into account the relationships among the characters—would be much easier. For instance, once the symbol for tree (ţţţ) is learned, the symbol for forest (ţţţţţ) and dense forest or jungle (ţţţţţţţ) are easy to remember. Consider also the task posed in Part II: learning the number 481216202428. Most people treat this number as a meaningless
string of digits—as unconnected facts that must be memorized by rote. To facilitate memorization, people often use memorization strategies such as packaging information into larger chunks (e.g., 481-216-202-428). Even so, consider how much easier the task would have been if the information had been presented in the following manner: 4-8-12-16-20-24-28. This presentation highlights the relationships among the digits—among the separate facts. Each successive element of the number can be determined by counting by four, which would be easy to remember. The traditional skills approach is like presenting the number 481216202428. By presenting facts, procedures, rules, and formulas in an unrelated manner, students are forced to memorize this information by rote, which makes learning school mathematics unduly difficult. A meaningful approach is like presenting the number 4-8-12-16-20-24-28. By underscoring the relationships among content, it makes learning school mathematics easier.

- **Meaningful learning makes remembering over the long term (retention) more likely.** People who memorize the Chinese characters or the number 481216202428 by rote are more prone to forget this information than those who learn this information in a meaningful manner. For example, memorizing 481-216-202-428 by rote works well for the short term, but could you recall the sequence tomorrow? Likewise, cramming for a test can help students pass the test but what are the chances they will remember the information after the test is over? On the other hand, meaningful learning (e.g., recognizing the relationship among the terms 4-8-12-16-20-24-28) makes it much more likely that a student will remember information even after the test is over.

- **Meaningful learning makes application (transfer) of knowledge more likely than memorization by rote.** Have students consider what would happen if the Number Lesson (Part II) was only the first of a series of number lessons and each lesson included twice as much information as the previous lesson (e.g., Lesson 2 involved learning 48121620242832364044485256)? Unfortunately, what they had learned in the first lesson would not help much in learning the new material in these subsequent lessons. Sooner or later, there would be just too much information to memorize by rote—no matter what memory tricks they used. Likewise, in the traditional skills approach, students—sooner or later—are overwhelmed with trying to memorize isolated bits of information. In contrast, meaningful learning enables students to use their understanding of relationships to learn new information or solve new problems. For example, recognizing the count-by-four pattern of the Number Lesson would make learning the second lesson (4-8-12-16-20-24-28-32-36-40-44-48-52-56) incomparably easier.

**Key for Probe 1.4** (pages 1-28 to 1-32)

**Part II**

2. a. The skills approach, exemplified by Mr. Ordin’s lesson, would be represented by point A.

b. The investigative approach, exemplified by Mrs. Perez’s lesson, would be represented by a point between points I and N. In this approach, a teacher gives students more freedom than either the skills or the conceptual approach, which would be represented by point H, but is more actively involved in planning instruction and guiding a class than in the problem-solving approach, which would be represented by point O. The latter is even more democratic because both teacher and students would propose problems and the class would vote on which it will explore. Although it is desirable to give students a voice in selecting problems, many times teachers need to exercise their own judgment about choosing a problem. In Part I of the probe, note that the teacher Mrs. Perez began her lesson with a problem chosen to help her students consider a specific mathematical concept (geometric progressions) and skills (procedures for computing multidigit multiplication). That is, the problem was posed to cover prescribed content, as well as vehicle for fostering mathematical thinking.

c. Y

d. H
Key for Investigation 1.4 (page 1-33)

Part I

1. An even number is one divisible by 2—that is, can be divided by 2 and leave no remainder. Informally, an even number can be defined as a collection that can be shared fairly (equally) between two people; an odd number, as a collection where fair sharing between two people would leave an item leftover.

2 & 3. A question that almost invariably comes up when students—even college-level students—play this game is: Is zero even, odd, or neither? If 0 is divisible by 2, then it meets the definition of an even number. This raises the question, What is 0 divided by a number—zero, undefined, or what? Note that many students have trouble remembering whether \( n \div 0 \) or \( 0 \div n \) is zero and which is undefined. In effect, can zero things be split into two groups with nothing leftover? The answer is yes. You would have two groups of nothing (zero) and nothing would be leftover. If this argument is too abstract for primary-level children, try an informal proof with a number line. Start with 12, say, and ask the class if the number is even or odd. After the children discover that 12 items can be divided into two groups evenly, have them consider, in turn, 11 to 1. Ask what pattern they notice and then ask what this pattern suggests about 0.

Beating the Odds. By setting up a table, it is clear that an odd player should put out one finger. This will give him or her a \( \frac{2}{3} \) chance of winning. (The odd player wins if the even player puts out zero or two fingers; the odd player loses if the even player puts out one finger.)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0(1/3)*</td>
<td>1(1/3)</td>
<td>2(1/3)*</td>
</tr>
<tr>
<td>1</td>
<td>1(1/3)</td>
<td>2(1/3)*</td>
<td>3(1/3)</td>
</tr>
<tr>
<td>2</td>
<td>2(1/3)*</td>
<td>3(1/3)</td>
<td>4(1/3)*</td>
</tr>
</tbody>
</table>

Based on the pattern above, children should recognize that 0 is an even number. Note that by extending the analogy further, older children can be helped to see that -2, -4, -6, and so forth can be considered even.

Extensions

The game is not fair, because the probability of an even player winning is \( \frac{5}{9} \). Why? Consider the following argument:\(^4\) The probability of the even player displaying zero fingers is \( \frac{1}{3} \); the same is true for the odd player. The probability of both displaying no fingers (a sum of 0), then, is \( \frac{1}{3} \times \frac{1}{3} \) or \( \frac{1}{9} \). The same logic can be applied to each of the other eight cells in the table below. The probabil-ity of any even outcome (indicated by an asterisk in the table below) is \( \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \) or \( \frac{5}{9} \).